The Bell Gedanken Experiment

Juan María Fernández

Physics Teacher

IES Jorge Manrique de Palencia (España)

juanmariafernandez@gmail.com

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Abstract

This document has two goals: 1) Give a brief introduction to the main ideas about the Bell's inequalities, with a Reference list. The inequalities and Bell's related work are perhaps the main theoretical advance on the meaning and interpretation of Quantum Mechanics from the controversy between Einstein and Bohr in the 1930s. This will be not an scholar article, but a simple description of the QM background and the Bell and D'Espagnat's ideas. 2) Describe **The Bell Gedanken Experiment**, an EJS simulator. If your main interest is using the simulator, you can go directly to the User Guide in section 5.This document is intended to be read on screen. It can be printed, but color and hyperlinks will be lost in that case.

If you find any error in the simulator or in this document, wrong concepts or language, please, feel free to use the above email and let my know.

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Contents

1 The Bell inequalities **3**

1.1 Proof of the Bell-Wigner-D'Espagnat inequality 3

2 The internals of the Bell-Wigner-D'Espagnat inequalities and the proof **4**

3 The system and the Classical and Quantum Mechanics probabilities **4**

4 Classical vs Quantum Mechanics **7**

4.1 The Bell-Wigner-D'Espagnat inequality in the Singlet Spin 1/2 system 9

4.2 Bell and D'Espagnat evaluation 9

5 User Guide **11**

5.1 Upper set of controls 11

5.2 Tabbed controls 12

5.2.1 QM_vs_CM controls 12

5.2.2 Bell_Test controls 12

5.2.3 Bell_Inequality controls 13

5.2.4 ViewCtrl controls 13

5.2.5 The Plots window 13

1 The Bell inequalities

There are many versions of the Bell inequalities, some from Bell itself and some from other authors, more or less refinated in scope and experimental utility[1]. The one presented here is the so called Wigner-D'Espagnat inequality, as quoted by Bell in [1][2]. It is perhaps the simpler, and is quite useful when discussing the perfectly anticorrelated Spin 1/2 system modeled in the EJS simulator presented here.

Wigner-D'Espagnat Inequality á la Bell: If A, B, C are three propositions about an arbitrary system

$$p(A, \overline{B}) + p(B, \overline{C}) \geq p(A, \overline{C}) \tag{1}$$

Where $\overline{B}$ is the negation of B, $\overline{C}$ is the negation of C, and $p(X, Y)$ is the probability of $\boldsymbol{X}$ **and**[3] $\boldsymbol{Y}$ being true.

1.1 Proof of the Bell-Wigner-D'Espagnat inequality

Because any probability is equal or greater than 0,

$$p(A, \overline{B}, C) + p(\overline{A}, B, \overline{C}) \geq 0 \tag{2}$$

with the obvious meaning of $p(X, Y, Z)$.

Adding to both sides of the equation $p(A, \overline{B}, \overline{C}) + p(A, B, \overline{C}) \geq 0$,

$$\left[p(A, \overline{B}, \overline{C}) + p(A, \overline{B}, C)\right] + \left[p(\overline{A}, B, \overline{C}) + p(A, B, \overline{C})\right] \geq \left[p(A, \overline{B}, \overline{C}) + p(A, B, \overline{C})\right] \tag{3}$$

For each square bracket we have

$$\left[p(A, \overline{B}, \overline{C}) + p(A, \overline{B}, C)\right] = p(A, \overline{B}) \tag{4}$$

$$\left[p(\overline{A}, B, \overline{C}) + p(A, B, \overline{C})\right] = p(B, \overline{C}) \tag{5}$$

$$\left[p(A, \overline{B}, \overline{C}) + p(A, B, \overline{C})\right] = p(A, \overline{C}) \tag{6}$$

This is the proof of equation (1).

[1] The versions, not the authors !
[2] Fron now on, you can click the hyperlinks.
[3] The logical AND.

2 The internals of the Bell-Wigner-D'Espagnat inequalities and the proof

Close examination of the proof shows that there are implicit hypotheses:

1. The system is a real thing, it does exists.
2. A,B,C and their negations are properties the system can have. This is to say: The system has, or not, that property enunciated in A,B,C, and it is possible to assign probabilities to each.
3. At least in principle, it is possible to elucidate in some way if the property so enunciated is owned or not by the system.

More deep considerations about that can be found in [1, 2, 3, 4].

3 The system and the Classical and Quantum Mechanics probabilities

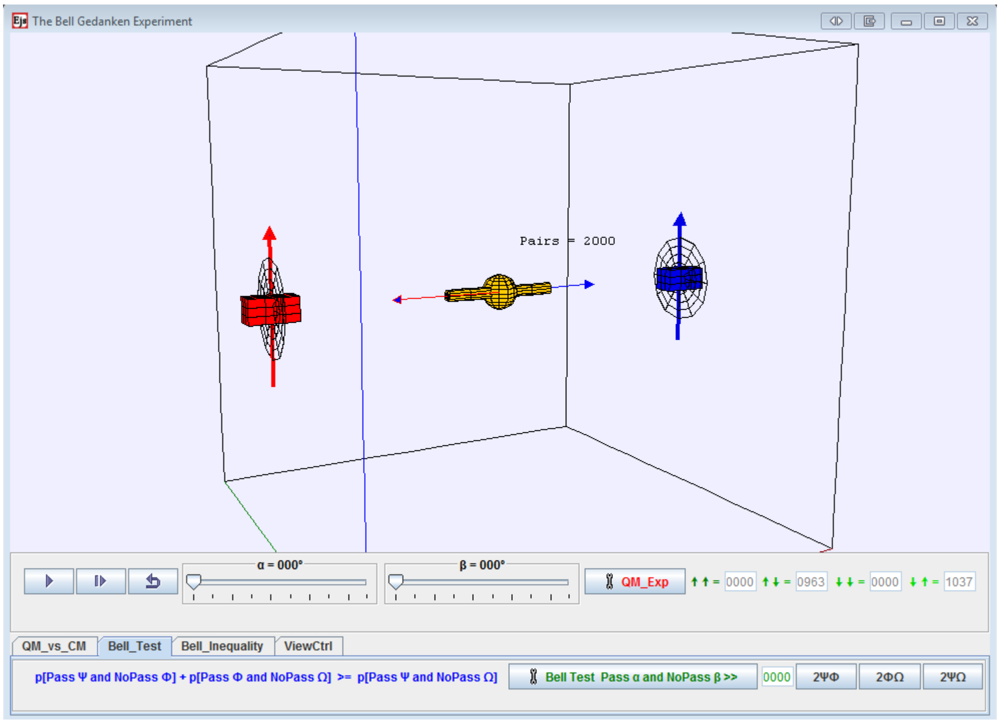

Figure 1: The singlet state Spin 1/2 system in the EJS simulator.

The system is a 3D simple version of the Bell's adaptation [1, 2] of the Einstein-Podolsky-Rosen [4] apparatus. A central SOURCE, orange, prepares a Singlet state for two Spin $\frac{1}{2}$ particles. The

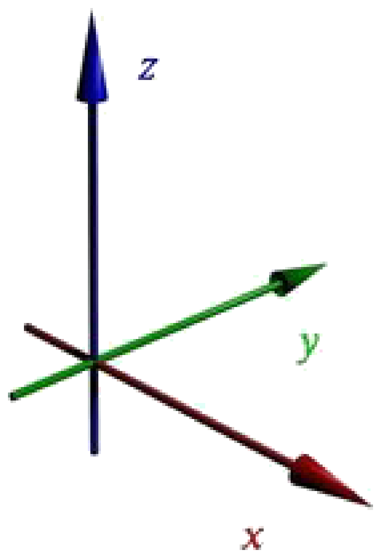

Figure 2: Axis convention .

Singlet is prepared for the z-component of the Spin. Each particle, the Blue and the Red, is send to oppossed Stern-Gerlach detectors, called BLUE and RED. The detectors and the source are aligned, and the distance detector-SOURCE is the same for BLUE and RED. The velocities of each particle are equal in modulus and opposite in direction. This way each particle arrives at same time to the correspondent detector. The detectors can be rotated about the y axis, and initially they both point in the positive z-direction. BLUE detector angle of rotation is α and RED's is β.

We take the angles and conventions of Cohen-Tannoudji et all [5] , where the axis system is shown in Figure (2). Each detector has a coordinate system, attached to its center. For BLUE detector positive y-axis going from SOURCE to BLUE, x positive is horizontal to the right, and z is the vertical axis, positive upwards. A similar and paralell coordinate system is used in RED detector, where the positive y axis goes from RED to SOURCE.

The Singlet state in the basis of the z component of the Spin, in standard notation of tensor product of bras and kets, and the above axis definition, is

$$|Singlet >= \frac{1}{\sqrt{2}} (|+>|->-|->|+>) \tag{7}$$

From now on, the first state in the tensor product refers to the Blue particle and the second to the Red. Following QM rules, the probability of having + - correlation is 50% , also 50 % for - + correlation, and 0 for + + and - - .

If the BLUE detector is rotated an angle α about the y axis, the new Spin base states along the new direction are[4]

[4]See the discussion for Spin 12/ and 2-Level systems in [5]

$$|+>_\alpha \;=\; cos\left(\frac{\alpha}{2}\right)\,|+>+sin\left(\frac{\alpha}{2}\right)|->\tag{8}$$

$$|->_\alpha \;=\; -sin\left(\frac{\alpha}{2}\right)\,|+>+cos\left(\frac{\alpha}{2}\right)|->\tag{9}$$

Inverting the matrix, the original states as a function of the new ones, are:

$$|+> \;=\; cos\left(\frac{\alpha}{2}\right)\,|+>_\alpha - sin\left(\frac{\alpha}{2}\right)|->_\alpha\tag{10}$$

$$|-> \;=\; sin\left(\frac{\alpha}{2}\right)\,|+>_\alpha + cos\left(\frac{\alpha}{2}\right)|->_\alpha\tag{11}$$

Supose RED detector in the $\beta = 0$ initial position, and the BLUE rotated as before; the Singlet state in the new base is

$$|Singlet>= \frac{1}{\sqrt{2}}\,[cos(\tfrac{\alpha}{2})\,|+>_\alpha|-> -sin(\tfrac{\alpha}{2})\,|->_\alpha|-> \tag{12}$$

$$-sin(\tfrac{\alpha}{2})\,|+>_\alpha|+> -cos(\tfrac{\alpha}{2})\,|->_\alpha|+>]$$

This way, the probabilities of finding $|+>_\alpha\,|->$,**Up Down**, or[5] $|->_\alpha\,|+>$, **Down Up**, are $\frac{1}{2}\,(cos\frac{\theta}{2})^2$, and the probabilites of finding $|+>_\alpha\,|+>$, **Up Up**, or[6] $|->_\alpha\,|->$, **Down Down**, are $\frac{1}{2}\,(sin\frac{\alpha}{2})^2$.

In case of BLUE detector rotating α and RED detector rotating β, both rotations about the y axis[7], it can be shown that the probabilities of finding[8] $|+>_\alpha\,|->_\beta$ or $|->_\alpha\,|+>_\beta$ are $\frac{1}{2}\,(cos\frac{\alpha-\beta}{2})^2$, and the probabilites of finding $|+>_\alpha\,|+>_\beta$ or $|->_\alpha\,|+>_\beta$ are $\frac{1}{2}\,(sin\frac{\alpha-\beta}{2})^2$. We also will use freely the notation **Up Down**, **Down Up**, **Up Up** and **Down Down**, for each one of the previous states.

State	**State also named**	**QM Probability**
$\|+>_\alpha\,\|->_\beta$	Up Down	$\frac{1}{2}\,(cos\frac{\alpha-\beta}{2})^2$
$\|->_\alpha\,\|+>_\beta$	Down Up	$\frac{1}{2}\,(cos\frac{\alpha-\beta}{2})^2$
$\|+>_\alpha\,\|+>_\beta$	Up Up	$\frac{1}{2}\,(sin\frac{\alpha-\beta}{2})^2$
$\|+->_\alpha\,\|->_\beta$	Down Down	$\frac{1}{2}\,(sin\frac{\alpha-\beta}{2})^2$

Table 1: QM probabilities in the Bell Gedanken Experiment.

[5]Not the logical OR !
[6]Again, not the logical OR.
[7]Along the lines SOURCE-BLUE detector, or RED detector-SOURCE
[8]It is possible to argue the same result from spatial isotropy and equation (12)

In Reference [1] are the Classical probabilites for the problem of finding the equivalent states in a Classical Mechanics-and-Electromagnetism context. The reader is encouraged to examine the whole article and thus the justification of the simple Classical problem, written in the Bell limpid style.

State	**State also named**	**Classical Probability**
$\|+>_\alpha \|->_\beta$	Up Down	$\frac{1}{2}-\frac{\|\alpha-\beta\|}{2\pi}$
$\|->_\alpha \|+>_\beta$	Down Up	$\frac{1}{2}-\frac{\|\alpha-\beta\|}{2\pi}$
$\|+>_\alpha \|+>_\beta$	Up Up	$\frac{\|\alpha-\beta\|}{2\pi}$
$\|->_\alpha \|->_\beta$	Down Down	$\frac{\|\alpha-\beta\|}{2\pi}$

Table 2: Classical probabilities in the Bell Gedanken Experiment.

There are strong differences between the Classical and the QM approach. In fact, only for angular differences of *0*, $\frac{\pi}{2}$ and π the Classical and QM values are the same.

4 Classical vs Quantum Mechanics

In this section we will follow loosely Bell in [1].

First , we assume the angular difference to be $\frac{\pi}{4}$[9]. The probabilities are

State	**State also named**	**Classical Probability**	**QM Probability**
$\|+>_\alpha \|->_\beta$	Up Down	0.375	0.427
$\|->_\alpha \|+>_\beta$	Down Up	0.375	0.427
$\|+>_\alpha \|+>_\beta$	Up Up	0.125	0.073
$\|->_\alpha \|->_\beta$	Down Down	0.125	0.073

Table 3: Classical and QM probabilities for $\alpha - \beta = \frac{\pi}{4}$ in the Bell Gedanken Experiment.

The difference between the Classical and QM approaches is evident.

[9] In The Bell Gedanken Experiment simulator, the angles are in degrees, for an easier reading.

Now we will test the difference between Classical and QM predictions using the Bell-Wigner-D'Espagnat inequality (1). It is useful to make explicit the hypotheses in the Classical description, with origin back in the Einstein-Podolsky-Rosen paper [4]. They defined the now called **Local Realistic Theories**. With some liberty taken in order and concepts, a such Theory foundations are:

1. The physical world has real and true existence, independent of any observer, and can be explored and be understood by reasoning and experiment.
2. It is possible, in special, to apply inductive reasoninig to establish Physical Laws.
3. The elements of the physical world do have properties associated to them.
4. If the theory assigns probability 1 to some value of a property, this property, and the value with probability 1, can be assigned to a system and be considered a real characteristic of the system.
5. Measurement processes are local. A measurement made in BLUE detector does not affect inmediately measurements made in RED detector and viceversa[10].

The above hypotheses have consequences for the Singlet Spin state in the system shown in Figure (1). Remember that SOURCE prepares the Singlet state with perfect anticorrelation in the value of the Spin component of the particles: If BLUE detector gets an **Up** value, RED detector gets a **Down** value, and viceversa, always. The Classical Local Realistic theory applied to this system makes the Spin Up or Down a property owned by each particle, the Blue and the Red. In the Einstein-Podolsky-Rosen idea, if QM cannot give exact predictions about the Spin value, and about the exact output of the detectors, is because QM is an incomplete theory, and there are some hidden varibles, not accounted in QM, that can be used in a new theory, to get rid of the statistical account and the probabilities in the actual incomplete QM description of the world.

The situation becomes more interesting when the detectors are rotated. Due to the exact anticorrelation, we know that if some measuring were made on the particles with both detectors rotated the same angle, one output would be Up and the other Down, always. Because the perfect anticorrelation does exists, even if we do not measure the other particle, say the RED, having measured the BLUE Spin in some direction, the Classical conclussion is that the RED has the opposite Spin value in that direction, with absolute certainty[11]. This way, to know, for example, the RED Sz and Sx Spin values, we measure the Sz on the RED particle, obtaining, let's say, Up, and then we measure on BLUE particle the Sx, obtaining Up. The perfect anticorrelation, the Local Realistic Theories and the Classical reasoning make us conclude that RED particle has, besides the Up value for Sz, a Down value for Sx.

[10] BLUE and RED can be changed by any two different places in space.

[11] Because a) the Spin value in some direction, in the Classical Local Theory, is a property the particle has from the very moment of being prepared in the SOURCE; b) the Singlet Spin state is a Total Spin 0 value, so each particle's Spin component value is opposite to the other.

4.1 The Bell-Wigner-D'Espagnat inequality in the Singlet Spin 1/2 system

This way, we can ask ourselves, in the laboratory, what are the probabilities of BLUE particles[12]

- i) Pass 0º and not Pass 45º
- ii) Pass 45º and not Pass 90º
- iii) Pass 0º and not Pass 90º

The Bell-Wigner-D'Espagnat inequality (1) is valid for any such Local Realistic Classical Theory, and for that reason

$$p(Pass\ 0^0\ , not\ Pass\ 45^0) + p(Pass\ 45^0\ , not\ Pass\ 90) \geq p(Pass\ 0^0\ , not\ Pass\ 90^0) \tag{13}$$

Taking in account the previous exposition, in the Classical Local Realistic Theory, in the perfect anticorrelated Singlet Spin system, the inequality is equivalent to

- i) BLUE Pass 0º and RED Pass 45º
- ii) BLUE Pass 45º and RED Pass 90º
- iii) BLUE pass 0º and RED PASS 90º

The QM probabilities for those outputs in the detectors are $\frac{1}{2}\,(sin\frac{\alpha-\beta}{2})^2$, and the Bell-Wigner-D'Espagnat inequality (13) becomes

$$\frac{1}{2}(sin(22.5^0))^2 + \frac{1}{2}(sin(22.5^0))^2 \quad \geq \quad \frac{1}{2}(sin(45))^2 \tag{14}$$

$$\Rightarrow 0.15 \quad \geq \quad 0.25 \tag{15}$$

And the inequalities in the ecs. (14) and (15) are not true.

4.2 Bell and D'Espagnat evaluation

It is clear from the previous section that Quantum Mechanics and the Classical Local Realistic Theories disagree in the experimental output predicted for Singlet Spin 1/2 systems[13]. What are the laboratory evidences? The Aspect et al. experiments, published in References [7, 8] are among the best considered, and they show strong evidence in favour of Quantum Mechanics. More recent experiments, as the Giustina et al. in Reference [9], with many others, reinforce the idea that

[12] We will use degrees for angles, in benefit of clarity, and the equivalences: a) Pass angle, get Up in the measurement with the detector rotated that angle, b) not Pass angle, get Down in the measurement with the detector rotated that angle.

[13] And in many other physical systems.

Local Realistic theories cannot describe properly the physical world, and Quantum Mechanics is the appropriate description. There are yet some experimental intrincacies to be resolved, but the great majority of physicist involved in this field are of this opinion.

In Quantum Mechanics the Spin values are not properties the particle owns, unless the prepared state is an observable's eigenstate. In general, the Spin gets its value in the measurement process, and QM can give only a probability of getting this or that value. The Singlet state, when the partices are far apart, is the cause of the entanglement of both particles. The state of the particles cannot be described independently, and the entanglement consequence is that when some Spin component value is measured in one particle, that makes inmediately the other particle of the pair getting the opposite value, if measured, and that happens be the distance between both particles a few milimeters or be the galaxy diameter. This "spooky action at a distance"[14] was, as is easily understood, very unpleasant to Einstein-Podolsky-Rosen, and to many more physicist.

This finish the theoretical introduction . More information can be obtained in the References.

[14] words attributed to Einstein.

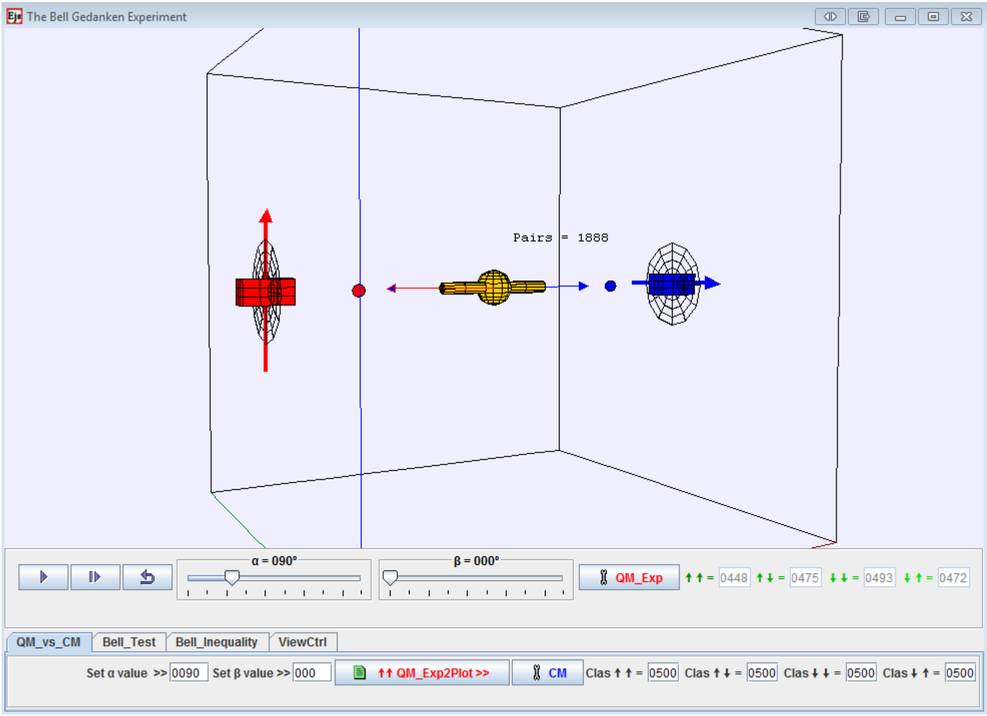

Figure 3: A 90º-0º run of the experiment .

5 User Guide

The **Bell Gedanken Experiment** is an EJS [6] java simulator created to explore the measurements in a colection of particles prepared in the Singlet Spin 1/2 state, and the comparison between Quantum and Classical Mechanics.The simulator has been prepared to recreate in graphical and easy way the operations, calculus and considerations made by Bell in Reference [1] and most of the D'Espagnat in Reference [2], as outlined in the preceding sections of this document.

When opening the simulator, you can see, as in the Figure (3) , the SOURCE emiting consecutive pairs of Blue and Red particles, prepared in the Singlet Spin 1/2 state. Then they arrive to the detectors, and the Spin measurements are made in the BLUE and RED detectors. Each run of the experiment has 2000 pairs of particles, and the number of pairs can be seen in the Laboratory 3D scene. The user can **click and drag** the 3D Laboratory view, to rotate the whole scene, while **Shift** + **click and drag** can zoom the whole scene.

5.1 Upper set of controls

- a) The Play, Step and Reinitialize buttons do what they are supposed to do.
- b) The α and β sliders change the angle of detectors. The rotation is about the positive y axis of each detector, as described in The System section (3), and the angles are measured with origin in the positive z-axis direction; α is the angle of BLUE detector, β is the angle of RED detector. Both angles are in degrees.

- c) The QM_Exp button makes a quick run of the experiment, whitout the particles animation, computing the pairs detected in each of the Up Up $\uparrow\uparrow$, Up Down $\uparrow\downarrow$, Down Down $\downarrow\downarrow$ and Down Up $\downarrow\uparrow$ states. The first state-arrow is for the Blue particle in the BLUE detector, and the second for the Red particle in the RED detector. The results are recorded in the next numerical boxes. The results are random, because the 2000 pairs of particles are randomly tested and measured in each of the detectors
- d) The four numerical boxes, to show the number of pairs measured for each of the above states. Each box is preceded by an explanatory label.These boxes are not editable.

5.2 Tabbed controls

5.2.1 QM_vs_CM controls

The **Quantum Mechanics** versus **Classical mechanics** set of controls are:

a) Set α value numerical box. Allow the fine seting of α angle for BLUE detector. Fractions of degree are not allowed.

b) Set β value numerical box. Allow the fine seting of α angle for RED detector. Fractions of degree are not allowed.

c) $\uparrow\uparrow$ QM_Exp2Plot button: To pass the $\uparrow\uparrow$ data in the Upper set of controls, to the Plot window (see below the Plots window description).

d) CM button: Makes the calculations of pairs in each of the Up Up $\uparrow\uparrow$, Up Down $\uparrow\downarrow$, Down Down $\downarrow\downarrow$, and Down Up $\downarrow\uparrow$ states, according to Classical Mechanics, with the α and β angles. The results are recorded in the next numerical boxes. The results in the boxes are the expected values of each pair, according to the Classical probabilities.

e) The four numerical boxes, to show the number of pairs calculated by d) above, in the Classical prediction.These boxes are not editable.

5.2.2 Bell_Test controls

a) First, there is a label with the Bell-Wigner-D'Espagnat inequality, for quick reference and nomenclature.

b) Bell Test Pass α and no Pass β button calculates how many Blue particles would Pass α and not Pass β. This is , as explained in section (4.1), equal to the number of Blue and Red pairs measured in the Up Up $\uparrow\uparrow$ state. The result appears in the next non editable greeen numerical box.

c) A geen numerical non editable box, showing the pairs calculated in d)

e) $2\Psi\Phi$ button. Pass the c) data to the first item data in the Bell-Wigner-D'Espagnat inequality. The result is shown in the Bell_Inequality Tab, described below.

f) $2\Phi\Omega$ button. Pass the c) data to the second item data in the Bell-Wigner-D'Espagnat inequality. The result is shown in the Bell_Inequality Tab, described below.

g) $2\Psi\Omega$ button. Pass the c) data to the third item data in the Bell-Wigner-D'Espagnat inequality. The result is shown in the Bell_Inequality Tab, described below.

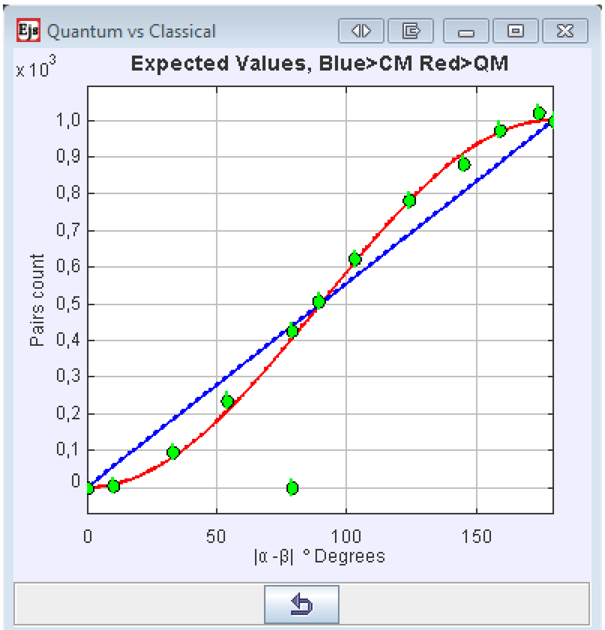

Figure 4: The Plots window .

5.2.3 Bell_Inequality controls

In this tab are shown the data computed and passed to each of the Inequality terms in the e), f) and g) buttons of the Bell_Test controls, described just above.

5.2.4 ViewCtrl controls

a) Zoom slider: to zoom the Laboratory scene.

b) Beep On/Off selector: to allow or not the beep sound in the detectors.

c) Show Plots and Show Laboratory radio buttons: allow to show or hide the Plots window and the Laboratory scene. When one is shown, the other is hidden. The Plots window is described in the next section.

5.2.5 The Plots window

The Plots window shows the data in a Up Up ↑↑ pair number vs angle difference representation. The Blue line is the Classical prediction for the expected value of pairs. The Red line is the QM prediction for the expected value of pairs and the Green points are the actual number of measured pairs in the experiment, shown in the Upper controls and passed by the ↑↑ QM_Exp2Plot button. Remenber that each run of the experiment has 2000 pairs.

The button in the window draws the Blue and Red curves.

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