Student difficulties with the number of distinct many-particle states for a system of non-interacting identical particles with a fixed number of available single-particle states

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We discuss an investigation of student difficulties with determining the number of distinct many-particle stationary states for a system of non-interacting identical particles. Here we focus on a system in which there are a fixed number of available single-particle states and a fixed number of particles but the total energy of the system is not fixed. The investigation was carried out in advanced quantum mechanics courses by administering free-response and multiple-choice questions and conducting individual interviews with students. We find that upper-level undergraduate and graduate students share many common difficulties related to these concepts. Many students struggled to determine the number of distinct many-particle stationary states and make connections between the number of distinct many-particle stationary states and the number of many-particle stationary state wavefunctions possible with the given constraints. Additionally, we found that many students had difficulty with mathematical sense-making in the context of quantum mechanics.

I. INTRODUCTION AND BACKGROUND

There have been a number of research studies aimed at investigating student reasoning in quantum mechanics (QM) [1–8]. However, there have been relatively few investigations into student difficulties with fundamental concepts involving a system of identical particles including the number of distinct many-particle states in a given situation.

Here we describe an investigation of upper-level undergraduate and graduate students' difficulties with determining the number of distinct many-particle stationary states for a system of non-interacting identical particles in which there are a fixed number of available single-particle states and a fixed number of particles but the total energy of the system is not fixed. In order to reason about these types of problems, students must not only have a good understanding of the basics of a system of identical particles but a strong background in combinatorics in order to determine the number of distinct many-particle states. However, prior studies suggest that students often struggle with mathematical sense-making and applying mathematical concepts correctly in the context of introductory physics even if they can solve isomorphic mathematical problems without the physics context [9].

Since human working memory while solving problems is restricted to a limited number of "chunks" and the size of a chunk in the working memory depends on the expertise of the individual who is solving the problems, Simon's framework of "bounded rationality" sheds light on the expert-novice differences in problem solving. This framework posits that individuals' rationality while solving problems is bounded by their expertise in that domain and they will make decisions while solving problems based upon their current level of expertise that satisfies them, but which may not be optimal [10]. Since students engage in sense-making which is commensurate with their current level of expertise, their integration of mathematical and physical concepts to solve a problem may not be appropriate for the problem solving task [10]. Moreover, some students may be motivated to find an optimal solution to the problem by searching for many alternative pathways in the problem space [10]. However, if their level of expertise is not sufficient to solve the problem on their own and they have not been provided with appropriate guidance and scaffolding support, they may experience cognitive overload and may not be able to correctly solve the problem posed [10, 11].

Before we discuss the common student difficulties in this context, we briefly review relevant concepts to solve the problems posed. In nature, there are two general types of particles: fermions with a half-integer spin quantum number (e.g., electrons and protons) and bosons with an integer spin quantum number (e.g., photons and mesons). A system of N identical particles consists of N particles of the same type (e.g., electrons). For a system of identical particles in classical mechanics (e.g., five identical tennis balls), each particle can be distinguished from all the other particles. In contrast, in QM, identical particles are indistinguishable and there is no measurement that can be performed to distinguish these identical particles from one another. For example, if the coordinate of two identical particles is interchanged, there is no physical observable that would reflect this interchange. For a system of identical fermions, it is not possible for two or more fermions to occupy the same single-particle state. For a system of identical bosons, it is possible for two or more bosons to occupy the same single-particle state.

Here, we will consider a system of identical particles in which the total number of particles is fixed. Also, for the systems considered here, the energy of the system is not fixed but there are only a fixed number of single-particle states available for the particles to occupy and there is no degeneracy in the single-particle energies.

In order to construct a many-particle stationary state for a system of fermions (ignoring spin degrees of freedom), there must be at least as many available spatial single-particle states as the number of identical fermions. If this condition is satisfied, one must determine the number of ways to arrange the identical fermions into the available single-particle states such that each single particle state has either zero or one fermion until all fermions have been placed into the available single-particle states. The number of ways to arrange N identical objects among M available slots ($M \geq N$) is $\binom{M}{N} = \frac{M!}{N!(M-N)!}$. Thus, for a system of N fermions with M available single-particles states, the number of distinct many-particle states is $\binom{M}{N}$ if $M \geq N$ and 0 if $M < N$.

One technique for determining the number of ways to arrange the identical bosons among the available single-particle states is often referred to as the "bin and divider" method. In particular, we can treat the single-particle states as bins to be filled with bosons and dividers to separate the different single-particle states, or bins. The number of distinct many-particle states can be found by determining the number of distinct arrangements. For a system of N identical bosons and M available single-particle states, there are $M-1$ identical dividers separating the single-particle states. This gives $N+M-1$ objects from which to choose the N identical bosons. Thus, the number of distinct many-particle states for a system of N indistinguishable bosons with M available single-particle states is $\binom{N+M-1}{N} = \binom{N+M-1}{M-1} = \frac{(N+M-1)!}{N!(M-1)!}$.

As a contrasting case, if identical particles could be treated as distinguishable, then one can determine which particle is in which single-particle state and there is no restriction on the number of particles in each single-particle state. For such a system with M available single-particle states, each particle can be placed in any of the M single-particle states. The number of distinct N-particle states for a system of N identical particles if they could be treated as distinguishable with M available single-particle states is M^N.

II. METHODOLOGY

Student difficulties with determining the number of distinct many-particle states for a system of identical fermions or bosons were first investigated using three years of data involving responses to open-ended and multiple-choice questions administered in class after traditional instruction in relevant concepts from 57 upper-level undergraduate students in a junior/senior level QM course and 30 graduate students in the second semester of the graduate core QM course. Additional insight was gained concerning these difficulties from responses of 14 students during a total of 81 hours of individual think-aloud interviews.

In all questions in our investigation, students were asked to consider the spatial part of the wavefunction to simplify the problem (i.e., students were asked to ignore the spin degrees of freedom). The word "identical" in this paper refers to all particles with the same properties and does not necessarily imply that the particles are indistinguishable since we also posed problems for a hypothetical case when the identical particles are distinguishable.

To probe whether students are able to determine the number of distinct many-particle states for a given system, the following question was one of six questions posed to the students. Question Q1 was posed to 30 graduate students and 25 undergraduate students on an open-ended in-class quiz after traditional lecture-based instruction on relevant concepts, as well as during the individual interviews.

Q1. *For a system of three non-interacting identical particles, there are four distinct single-particle states $\psi_{n_1}(x)$, $\psi_{n_2}(x)$, $\psi_{n_3}(x)$, and $\psi_{n_4}(x)$ available to each particle. How many different three-particle states can you construct if the particles are (a) fermions (Ignore spin), (b) bosons? (Ignore spin), or (c) distinguishable particles? (Ignore spin).*

TABLE I. The percentages of graduate (N=30) and undergraduate (N=25) students who correctly answered question Q1.

Type of Particle	Graduate (%)	Undergraduate (%)
Fermions	40	48
Bosons	17	16
Distinguishable	17	20

In Q1(a), for a system of three identical fermions, there are four distinct single-particle states in which to place the three fermions. Since no single-particle state can have more than two fermions, there are $\binom{4}{3} = \frac{4!}{3!(4-3)!} = 4$ distinct three-particle states in Q1(a). For the system of three identical bosons, a single-particle state can have more than one boson. In Q1(b), there are $\binom{4+3-1}{3} = \binom{6}{3} = \frac{6!}{3!(6-3)!} = 20$ distinct three-particle states for a system of three identical bosons. In Q1(c), for the contrasting case of identical particles that can be treated as distinguishable, each particle can be placed in any of the four single-particle states. Since there is no symmetrization requirement and the particles are distinguishable, each two-particle state is distinct. There are $4^3 = 64$ distinct three-particle states for a system of three identical particles that can be treated as distinguishable.

III. STUDENT DIFFICULTIES

Many students struggled to determine the number of distinct many-particle states for a system of identical particles. Table I lists the percentages of students who answered question Q1 correctly for a system of three identical particles after traditional instruction. We will discuss several categories of student difficulties that interfered with their ability to determine the number of distinct many-particle states for a system of non-interacting identical particles. These categories include (1) conceptual difficulties, (2) reliance on memorized formulas, and (3) difficulty with procedural knowledge. Some of the difficulties discussed here may be placed in several categories, but we listed them in a particular category in an effort to illustrate these broader categories with explicit examples. Some students provided the same incorrect answer to Q1 but reasoned about them differently based upon different underlying difficulties and therefore, the same incorrect answer may be in different categories based upon the reasoning provided by the students (note that the categories are not mutually exclusive). We used the interview to probe deeper into these difficulties and better understand the underlying reasoning behind the common incorrect answers.

A. Conceptual difficulties with indistinguishability pertaining to a system of identical particles: A system of identical fermions or bosons consists of indistinguishable particles and one must be careful not to count the number of many-particle states as if they are distinguishable particles. For example, if two identical fermions are in the single-particle states labeled by ψ_{n_1} and ψ_{n_2}, then there is no way to distinguish the system in which the first fermion is in the single-particle state ψ_{n_1} and the second fermion is in the single-particle state ψ_{n_2} from the system in which the first fermion is in the single-particle state ψ_{n_2} and the second fermion is in the single-particle state ψ_{n_1}. These two arrangements of the fermions make up the different terms of a two-particle stationary state wavefunction and are not two distinct two-particle states. Assuming $n_1 \neq n_2$, the completely antisymmetric two-particle wavefunction for this system (ignoring spin) is $\frac{1}{\sqrt{2}}[\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_1}(x_2)\psi_{n_2}(x_1)]$.

Many students struggled to identify how the number of distinct many-particle states for a system of indistinguishable particles would differ from that of a system of identical particles that can be treated as distinguishable. Roughly one-fourth of the graduates and one-tenth of the undergraduates incorrectly determined the same number of distinct many-particles states in Q1 for a system of indistinguishable fermions or bosons as for a system of identical particles that can be treated as distinguishable.

For a system of identical fermions, the most common incorrect answer to Q1(a) was $4 \cdot 3 \cdot 2 = 24$ distinct many-particle states. Students with this response claimed that there were 4 single-particle states available to place the first particle, 3 available single-particle states for the second particle, and 2 remaining available single-particle states for the last particle. One interviewed student with this type of response claimed that "there are four states for the first fermion, three for the second (fermion) since it cannot be in the same state as the first (fermion), and two left for the last one." This student and others with similar reasoning often correctly tried to apply Pauli's exclusion principle but did not take into account the fact that these three fermions are indistinguishable. By way of example, if the three fermions occupy the single-particle states ψ_{n_1}, ψ_{n_2}, and ψ_{n_3} there is no way to detect which fermion is in which single-particle state. Assuming $n_i \neq n_j \neq n_k$, the many-particle stationary state wavefunction for this system is $\Psi(x_1, x_2, x_3) = \frac{1}{\sqrt{6}}[\psi_{n_i}(x_1)\psi_{n_j}(x_2)\psi_{n_k}(x_3) - \psi_{n_i}(x_1)\psi_{n_k}(x_2)\psi_{n_j}(x_3) + \psi_{n_j}(x_1)\psi_{n_k}(x_2)\psi_{n_i}(x_3) - \psi_{n_j}(x_1)\psi_{n_i}(x_2)\psi_{n_k}(x_3) + \psi_{n_k}(x_1)\psi_{n_i}(x_2)\psi_{n_j}(x_3) - \psi_{n_k}(x_1)\psi_{n_j}(x_2)\psi_{n_i}(x_3)]$. There are 4 ways to choose the labels n_i, n_j, and n_k from the available states labeled by n_1, n_2, n_3, and n_4. If the particles could be treated as distinguishable, the six terms in this completely antisymmetric many-particle stationary state would be six distinct many-particle states producing a total of 24 distinct many-particle states, which is what the students in Q1(a) reasoned was the case for fermions.

For a system of identical bosons, many students struggled to identify how the number of distinct many-particle states would differ from that of a system of identical particles that can be treated as distinguishable. In Q1(b), students who claimed that there were $4^3 = 64$ distinct many-particle states for a system of indistinguishable bosons often stated that since bosons can occupy the same single-particle state, there are 4 available single-particle states for each boson and thus, $4 \cdot 4 \cdot 4 = 64$ distinct many-particle states. However, these students were not taking into account the fact that the bosons are indistinguishable and therefore some states that are distinct for a system of distinguishable particles are not distinct for a system of indistinguishable bosons. For example, students with this type of reasoning incorrectly counted the many-particle states $\psi_{n_1}(x_1)\psi_{n_1}(x_2)\psi_{n_2}(x_3)$, $\psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_1}(x_3)$ and $\psi_{n_2}(x_1)\psi_{n_1}(x_2)\psi_{n_1}(x_3)$ as distinct many-particle states in Q1(b). Instead, these three states correspond to the three terms in a completely symmetric many-particle stationary state wavefunction $\frac{1}{\sqrt{3}}[\psi_{n_1}(x_1)\psi_{n_1}(x_2)\psi_{n_2}(x_3) + \psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_1}(x_3) + \psi_{n_2}(x_1)\psi_{n_1}(x_2)\psi_{n_1}(x_3)]$ and are not three distinct many-particle states for the system of indistinguishable bosons.

In addition, some students struggled to determine the number of distinct many-particle states in part because they had difficulty realizing that the order in which the single-particle wavefunctions are expressed in the product of the single-particle states in the many-particle basis states is irrelevant. For example, the following are all equivalent ways to express a basis state for a system of three non-interacting identical particles: $\psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_3}(x_3)$, $\psi_{n_1}(x_1)\psi_{n_3}(x_3)\psi_{n_2}(x_2)$, $\psi_{n_2}(x_2)\psi_{n_1}(x_1)\psi_{n_3}(x_3)$, $\psi_{n_2}(x_2)\psi_{n_3}(x_3)\psi_{n_1}(x_1)$, $\psi_{n_3}(x_3)\psi_{n_1}(x_1)\psi_{n_2}(x_2)$, and $\psi_{n_3}(x_3)\psi_{n_2}(x_2)\psi_{n_1}(x_1)$. However, in Q1, students with this type of difficulty counted each of these equivalent products of the single-particle states as if it were a distinct many-particle state for the system. Some students focused on the order in which the labels for the single-particle states or the coordinates appeared to determine whether the products of the single-particle wavefunctions was different from each other. For example, when comparing $\psi_{n_1}(x_1)\psi_{n_2}(x_2)\psi_{n_3}(x_3)$ and $\psi_{n_1}(x_1)\psi_{n_3}(x_3)\psi_{n_2}(x_2)$, they claimed that n_2 and/or x_2 appear in the second place in the first product and in the third place in the second product, so these must be different terms in the many-particle wavefunction.

B. Reliance on memorized formulas: Over 60% of the interviewed students struggled in Q1 and used a memorized formula for the number of many-particle states rather than formulating a systematic reasoning for the given system. For example, whether a student would reason about a given situation to find the number of distinct many-particle states conceptually or use memorized knowledge depended on the context. Some students could identify that two or more fermions could not occupy the same single-particle state in one context but not in a different situation. For example, during the interview, students were asked to write all of the possible many-particle stationary state wavefunctions for a system of three indistinguishable fermions in two distinct single-particle states and then later asked to determine the number of distinct many-particle states for this same system. It is not possible to have three fermions in only two single-particle states. However, some students incorrectly provided at least one many-particle stationary state and/or calculated a non-zero number of distinct many-particle states for this system. Generally, these students either determined that there are zero distinct many-particle states for a system in which two or more fermions are in the same single-particle state or that the many-particle stationary state wavefunction does not exist for such a system, but then answered the other question as though such a system does exist. One interviewed student with this difficulty correctly stated that "we can't write a many-particle stationary state wavefunction for a system that has two fermions in the same state." But this same student later in the interview incorrectly calculated that there are $\binom{3}{2} = 3$ distinct many-particle states for the system of three indistinguishable fermions in two single-particle states. Interviews suggest that students with this type of difficulty often used the formula $\binom{M}{N}$ from memory to find the number of distinct arrangements, but they did not do a reasonability check for whether this formula should be used for the given situation and correctly identify what N and M represent. For example, upon questioning by the interviewer, this interviewed student incorrectly identified M as the number of identical fermions as opposed to the number of available single-particle states and incorrectly identified N as the number of available single-particle states as opposed to the number of fermions. He did not detect the inconsistency in his two responses. The lack of sufficient sensmaking to recognize that different responses for the same physical system may be inconsistent is common in introductory physics but has also been observed in prior research related to student understanding of Dirac notation in QM [6, 8]. One reason for the difficulty in this type of metacognition is that students' working memory may be constrained by the demands of the problem and the cognitive overload may make it difficult to do metacognition and ensure that different responses are consistent with each other [11].

C. Difficulties with procedural knowledge: More than one-third of the interviewed students who were able to correctly determine the number of distinct many-particle states for a system with a small number of particles and available single-particle states had difficulty generalizing to a system with a large number of particles and available single-particle states. Many of them used their intuition and deductive reasoning after determining the first few distinct many-particle states to calculate the total number of distinct many-particle states. This was particularly true for a system of indistinguishable bosons in Q1(b). In general, students were able to correctly calculate all the distinct three-particle states when the three bosons were in the same single-particle state. Then they explicitly listed the first few distinct three-particle states when all three bosons were in different single-particle states and attempted to identify a pattern to enable them to calculate the total number of distinct many-particle states. Often, students applied a similar tactic to determine the total number of three-particle states when two of the bosons are in one single-particle state and one boson is in a different single-particle state. However, some students incorrectly generalized their pattern and miscounted the total number of distinct three-particle states in Q1(b). Other students who answered Q1(b) correctly had difficulty generalizing to a system with a large number of particles and available single-particle states. In general, they were able to explicitly list all of the possible many-particle states for a system with relatively few particles and available states, but struggled to recognize a pattern to generalize the results correctly. For a system with a large number of particles and a large number of available single-particle states, they could not list every possible many-particle state and were not able to calculate the number of distinct many-particle states.

IV. SUMMARY AND FUTURE PLAN

Our investigation suggests that upper-level undergraduate and graduate students have many common difficulties with fundamental concepts involving a system of non-interacting identical particles. The student difficulties discussed here can be interpreted using Simon's bounded rationality framework (in that students who are developing expertise are limited in their cognitive resources when solving these types of QM problems [10]) and Sweller's cognitive load framework (in that if appropriate scaffoldng support commensurate with students' current level of expertise is not provided, students will experience cognitive overload [11]). We find that since the students are still developing expertise in these concepts, they often only attempted a solution that was satisfactory to them in which they saw no inconsistencies even though there were inconsistencies based upon expert cognitive task analysis of the problems [12]. Interviews suggest that some students recognized that their initial solution plan may not be optimal but without sufficient guidance and scaffolding support, cognitive overload in this novel domain in which they are still developing expertise prevented them from contemplating better alternative pathways in the problem space to solve the problem correctly [10, 11]. Since the paradigm of QM is novel, these issues become critical and must be taken into account in developing curricula and pedagogies to help students learn these concepts. We are using the difficulties found in this context a guide to develop and validate a Quantum Interactive Learning Tutorial (QuILT) that strives to provide appropriate guidance and scaffolding support to help students develop a solid grasp of these fundamental concepts. The initial in-class evaluation of the validated QuILT is encouraging.

ACKNOWLEDGEMENTS

We thank the NSF for awards PHY-1505460 and 1806691.

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