Algebra-based students & vectors: assessing physical understanding in arrow vs *ijk*

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A recent study of calculus-based introductory physics students found that they performed significantly better on vector addition and subtraction tasks when the questions were given using the ijk representation instead of an "arrows-on-a-grid" representation, and also presented evidence that working knowledge of the ijk format was necessary to correctly perform vector operations in the arrow representation. A subsequent study found that algebra-based physics students also performed significantly better in the ijk representation than the arrow representation in both one- and two-dimensional problems, even though no explicit ijk instruction was given in the course. In this follow-up investigation we asked algebra-based physics students to reason physically about their answers, in order to assess if the higher performance on ijk questions indicates physical understanding or is the result of algorithmic "plug-and-chug" thinking. While students using ijk were more successful in computing the correct answer, their use of physical reasoning in justifying their answers was not significantly different than those using the arrow format. However, the higher number of students both correctly calculating and reasoning in the ijk format suggests that these students may benefit from ijk instruction.

I. INTRODUCTION

Student difficulties with vector addition and subtraction in the arrow representation have been well documented [1–8]. In addition to finding vector subtraction more difficult than addition, prior work has found: the relative position and context of vectors can cue a particular solution (or error) [5, 6], students tend to stick with their initial solution method when given different visual vector representations [7], and student performance on one-dimensional vector subtraction is affected by the vectors' relative orientation [8].

In addition to studies primarily focused on student difficulties with vectors in the arrow representation, there have been studies comparing performance on vector tasks across representations, typically the arrow and ijk representations [9, 10]. Heckler and Scaife were the first to do this systematically, comparing how students in a calculus-based introductory physics course performed on vector tasks in both representations [9]. They found that students performed better on ijk tasks than arrow tasks, particularly for vector subtraction, and found no significant difference in the ability to reason physically about their answers. Evidence of a knowledge hierarchy was also observed: success in the ijk representation was necessary for success in the arrow representation.

We conducted a follow-up study comparing the performance of students in an algebra-based introductory course on multiple-choice ijk and arrow vector tasks [10]. Our results were consistent with those in Ref. [9]; students consistently performed better on tasks in the ijk representation than in the arrow representation. This was unexpected since no explicit ijk instruction was given in the algebra-based course. Indeed, many popular "College Physics" texts omit the ijk format [11]. While vector components are used (A_x, $\mathbf{A}_x$, or $\vec{A}_x$), writing out vectors in terms of unit vectors is not present.

This paper extends our prior work in two primary ways: asking students to answer a free-response question rather than multiple-choice, and asking students to explain if their answers make physical sense. Changing the question from multiple choice to free-response allows students to choose any solution path they wish, and asking them to reason physically about their answer should allow us to see if the higher rate of correct responses in the ijk format are the result of "plug-and-chug" thinking. In this study, "reason physically" means associating the direction of the computed vector with the direction of the average force exerted on the object.

II. METHODS

Students participating in the study were in a large-enrollment (300 students) algebra-based introductory mechanics course during the fall of 2015. All students met in the same lecture section in a large lecture hall for 50 minutes three times a week. The lecture used a mix of traditional and Peer Instruction methods, with weekly online assignments [12]. The associated lab course is not required for the lecture, though 82% of students in the study were also enrolled in lab. The author of this paper was the instructor for the lecture section under study.

The assessment, hereafter referred to as the Momentum Vector Task (MVT), was given in class (10 minutes) on a handout for a small amount of extra credit during the last week of class. Students were given the option to opt out of the study but still receive credit for completing the handout. Students were randomly assigned either the "arrows-on-a-grid" version ($N = 130$) or the ijk version ($N = 126$) of the MVT.

The MVT consisted of a single free-response 2D vector subtraction question in the context of Newton's 2nd Law. To better compare with the results of Heckler and Scaife, we used the question given in Fig. 13 of Ref. [9], which is shown in Figs. 1 and 2. Students were given the initial and final momenta of an object and asked to find the average force exerted on the object if the momentum change occurred over 1s. Students were then asked if the direction of the average

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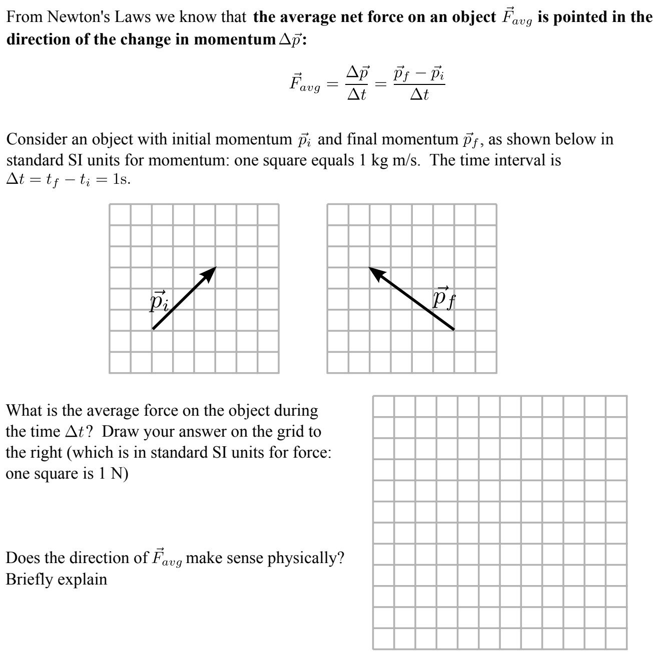

FIG. 1. "Arrow" version of the MVT. In the actual handout, additional whitespace was present for student responses. The question was taken from Fig. 13 of Ref. [9].

force they found made sense physically, and to explain their answer.

Student responses were categorized in two ways: whether or not the correct average force was found, and if they used correct physical reasoning to justify the direction of the average force. To qualify as "correct" in the ijk format, students must have explicitly used components either via $\hat{i}$ and $\hat{j}$ unit vectors or writing out the vectors as an ordered pair such as $p_f = (-4, 3)$ (though no students did the latter). In the arrow representation, students needed to draw the correct vector and show some evidence that they were using vector subtraction correctly, such as drawing the arrow subtraction explicitly.

To categorize "physical reasoning", we adopted the criteria in Ref. [9]: "responses that provided at least some significant and physically correct insight into the solution." To satisfy this, students needed to indicate that they were connecting the direction of the average force to the direction of the change in momentum. Following Heckler and Sciafe, we were somewhat liberal in categorizing these responses, and responses in the other category were either completely wrong or did not connect force to momentum in any meaningful way.

III. RESULTS

Table I summarizes the results on the MVT. The percent of students correctly computing the answer in the ijk representation (66%) was significantly higher than those answering correctly in the arrow representation (25%) [$t(254) = 7.26, p < 0.001, d = 0.9$]. The percent of students who provided fully or partially correct physical reasoning was not significantly different between the ijk (44%) and arrow (40%) representations [$\chi^2(1) = 0.35, p = 0.55$].

The breakdown of physical reasoning responses for the subset of students who correctly found the average force is given in Table II. Of the students finding the average force correctly, the percent providing fully or partially correct physical reasoning in the ijk format (52%) is *lower* than the arrow format (69%), though the difference is only marginally significant [$\chi^2(1) = 2.70, p = 0.10, d = 0.34$]. However, the "medium" effect size of 0.34 may be of instructional interest. Table III gives examples of physical reasoning responses categorized as (at least partially) correct or incorrect.

From Newton's Laws we know that **the average net force on an object $\vec{F}_{avg}$ is pointed in the direction of the change in momentum $\Delta\vec{p}$:**

$$\vec{F}_{avg} = \frac{\Delta\vec{p}}{\Delta t} = \frac{\vec{p}_f - \vec{p}_i}{\Delta t}$$

Consider an object with initial momentum $\vec{p}_i$ and final momentum $\vec{p}_f$, as shown below in standard SI units for momentum: kg m/s. The time interval is $\Delta t = t_f - t_i = 1$s.

$$\vec{p}_i = 3\hat{i} + 3\hat{j} \qquad \vec{p}_f = -4\hat{i} + 3\hat{j}$$

What is the average force on the object during the time Δt? Write your answer below (assume the answer is in standard SI units for force (N)):

Does the direction of $\vec{F}_{avg}$ make sense physically? Briefly explain.

FIG. 2. "ijk" version of the MVT. In the actual handout, additional whitespace was present for student responses. The question was taken from Fig. 13 of Ref. [9].

TABLE I. Number and percent of correct and incorrect responses for finding the average force ("Correct?") and the presence of at least partially correct physical reasoning ("Reasoning?") .

	Correct?		Reasoning?	
	Y	N	Y	N
ijk ($N = 126$)	83 (66%)	43 (34%)	55 (44%)	83 (66%)
Arrow ($N = 130$)	32 (25%)	98 (75%)	52 (40%)	78 (60%)

TABLE II. Number of responses showing fully or partially correct physical reasoning for the subset of students who correctly computed the average force.

	Reasoning?	
	Y	N
ijk ($N = 83$)	43 (52%)	40 (48%)
Arrow ($N = 32$)	22 (69%)	10 (31%)

IV. DISCUSSION

The difference across representations in the number of students who correctly answered the vector subtraction question was consistent with earlier work [9, 10], with the number of correct responses in the ijk format (66%) significantly higher than those in the arrow format (25%), shown in Table I . While the trend is similar, we note that these percentages are lower than those from the analogous multiple-choice task we used in an earlier study given to students in a different semester of the same course [10]. In the task used in the earlier study, the average score for 2D-subtraction questions was 76% in the ijk format and 59% in the arrow format. While we did not test this explicitly, it is reasonable that the multiple-choice format of the earlier study artificially inflated scores, allowing students to recognize a correct answer even if they could not recall it themselves. Despite the lower rate of correct responses in this study, students still responded correctly more often in the ijk format, even though no explicit ijk instruction had been provided in lecture.

Overall students connected the direction of the average force with some aspect of the change in momentum at equal rates in both the ijk (43%) and arrow formats (40%), shown in Table I. While the percentages are lower, these results are consistent with the results from Heckler and Sciafe, the percent of responses showing fully or partially correct physical reasoning was similar in both the ijk (60%) and arrow (49%) representations [9]. We note that these percentages include those who found an incorrect answer for the average force, but still connected this direction with the direction of the change in momentum.

The most common incorrect answer in the arrow format was consistent with adding the vectors rather than subtracting them, giving a vector that points "up and left" ($N = 33$, 25%). Students provided reasoning such as number 8 in Table III; "Yes its final momentum was greater in the x direction in the negative than its initial so the object wanted to go back." While this response does not account for the vertical direction (and is not quantitatively correct), these types of responses were counted as "partially correct" since we felt that counting these answers as "incorrect" would artificially suppress the amount of physical reasoning students were using in the arrow representation.

The most common incorrect answer in the ijk format was to simply *ignore* the unit vectors completely, directly combining components together such as "$-4+3-(3+3) = -7$" ($N = 12$, 10%). One example of this error (with correct physical reasoning) is given by number 4; "No, the object would be going in the opposite direction it started in so it would make sense that the force would be in the negative direction as well." This particular student ignored the $\hat{i}$, $\hat{j}$ and computed $(\vec{p}_i - \vec{p}_f)$, rather than $(\vec{p}_f - \vec{p}_i)$. Still, the student realized that their answer of a positive force did not make sense and so the answer was counted as "partially correct" for physical reasoning. Additional examples of reasoning that was counted as "correct" for each representation is given in Table III.

Among those who correctly computed the average force, the percent of students in each representation using fully or partially correct physical reasoning was not statistically different between the arrow (22 students, 69%) and ijk (43 students, 52%) representations, shown in Table II. This is consistent with the results of Heckler and Scaife for students in a calculus-based course where students had received explicit instruction in both the arrow and ijk formats [9]. The lack of statistical significance is a result of the small sample size for those who correctly answered the question in the arrow format. Despite this, the percent of correct answers which had accompanying physical resonating was much higher in the arrow representation (69%) than the ijk representation (52%), with a "medium" effect size ($d = 0.34$). This is flipped from what was observed for students in the calc-based course; students who correctly found the average force were more likely to have correct physical reasoning in the ijk format (70%) than in the arrow format (58%) [9].

One possible explanation for the observed differences between the algebra-based students in this study and the calc-based students in Heckler's study is that students in this study were not given any explicit instruction in the ijk method, but had received explicit instruction in the arrow method. Without knowing what $\hat{i}$ and $\hat{j}$ actually were, students who correctly computed the average force may have been unsure of how to interpret it. Examples of student responses in this category are: "Unknown, since I am unable to define $\hat{i}$," "No because I have no idea what the $\hat{i}\&\hat{j}$ means, therefore $-7\hat{i}/1$s doesn't make any sense as an answer," and "No since I don't know what $\hat{i}\&\hat{j}$ is." Perhaps the most impressive answer in this category was the following; "I was not sure what $\hat{i}$ and $\hat{j}$ stood for so I made them my axis, since [the answer is] only in [the] $\hat{i}$ [direction, the vector] must only be left or right, since [the vector is] negative [it] has to be left." Here the student explicitly acknowledged that they did not know what $\hat{i}$ and $\hat{j}$ meant, but they were compelled to interpret them as directions, even including a correct picture. Explicit ijk instruction may alleviate this particular issue.

In addition to students who explicitly mentioned not knowing how to interpret $\hat{i}$ & $\hat{j}$, many students manipulated the unit vectors correctly in their computation but did not interpret the resulting vector direction appropriately. There were 10 students who correctly found $\vec{F}_{avg} = -7\text{N}\hat{i}$, but were concerned that the force was "a negative number", saying that "vectors can't be negative." It may be they are misinterpreting the negative component they found as a negative magnitude, and thus rejecting their answer. Again, with explicit ijk instruction, the number of these responses might be lessened.

Taken together, the larger (though marginally significant) percent of correct physical reasoning for students who correctly computed the force in the arrow MVT (69%) than the ijk MVT (52%) and the accompanying student explanations support the claim that students do not necessarily understand what they are doing any more in the ijk format than the arrow format, despite the higher ijk scores. However, the nature of those student explanations and the fact that the percent of

TABLE III. Responses counted as at least partially correct for "physical reasoning" question. "$\vec{F}_{avg}$?" indicates if the student correctly found the average force ("Y") or not ("N").

#	Rep.	$\vec{F}_{avg}$?	
1	ijk	Y	"Yes because the final momentum is in the negative direction"
2	ijk	Y	"Yes, if the change in momentum is to the left, since the $\vec{F}_{avg}$ is negative. If it was to the right, $\vec{F}_{avg}$ would be positive"
3	ijk	N	"Yes because the second momentum kills all y-direction force so it only moves in the x-direction"
4	ijk	N	"No, the object would be going in the opposite direction it started in so it would make sense that the force would be in the negative direction as well."
5	Arr	Y	"Yes, the two vertical forces cancel and you are left with a negative force in the horizontal direction."
6	Arr	Y	"Yes, the initial momentum is pointing upward & to the right, in order to cause the momentum to be upward & left, there should be a force acting to the left that is 7 kg m/s."
7	Arr	N	"It does not, because I would think that it would be a negative force since the vector changes direction?"
8	Arr	N	"Yes its final momentum was greater in the x direction in the negative than its initial so the object wanted to go back"

students who provided both correct answers and physical reasoning in the ijk format (34%) was double that of the arrow format (17%) suggests that, at least in this case, students in an algebra-based introductory physics course would benefit from additional vector instruction in the ijk format.

V. CONCLUSIONS

The primary results of this work show that students in an algebra-based introductory physics course correctly answer free-response 2D vector subtraction questions more often in the ijk format than the arrow format, but justify their answer with fully or partially correct physical reasoning at the same frequency in both formats. This is consistent with earlier work showing that students in an algebra-based course perform better on multiple choice 2D vector subtraction questions when given in the ijk format [10], and that students in a calculus-based course provide fully or partially correct physical reasoning at roughly equal rates in either format [9]. The successful performance of the algebra-based students in the ijk format is perhaps surprising since no explicit instruction on ijk was given in the course, and the ijk format is typically absent from many popular texts aimed at this audience [11]. While there was some evidence that the higher ijk scores did not indicate a better physical understanding, the nature of the student explanations suggest that augmenting the typical vector instruction in an algebra-based course with ijk instruction would have some benefits. Students in the algebra-based course were less successful at completing vector tasks than the calculus-based students in Ref. [9], consistent with results in Ref. [2], where algebra-based students were less successful at vector tasks that calculus-based students.

Future work will investigate if the increased performance in the ijk format can be used to prime students to perform better, both with calculations and with physical reasoning, in the arrow format. Evidence of this effect has already been found for lower-performing students in a calculus-based introductory physics course [9], and we aim to replicate that study with students in the algebra-based course.

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