Tracing Difficulties With Relativistically Invariant Mass To Difficulties With Vector Addition Of Momentum In Newtonian Contexts

Andrew Boudreaux

Western Washington University
Department of Physics and Astronomy
Bellingham, WA 98225-9064
boudrea@physics.wwu.edu

Abstract. For effective transfer of knowledge, it is necessary to break the transfer of conceptual difficulties. In physics courses that include special relativity, students are expected to relate the invariant mass of a system to the energy and momentum of the individual particles that make it up. Many have difficulty doing so. Student responses indicate that some difficulties stem from a failure to treat energy-momentum as a four-vector. Introductory students experience related difficulties in a purely non-relativistic context: many fail to take the vector nature of three-momentum into account when relating the momentum of a system to the momenta of its constituents. Results suggest that these difficulties are widespread, and not necessarily resolved through the study of advanced topics.

INTRODUCTION

Many instructors – in physics and other disciplines – expect their students to be able to transfer ideas they have learned to new situations. Research on student learning in a variety of domains indicates, however, that this ability is often not a result of instruction.[1,2] An experienced physics instructor will recognize that introductory students encounter difficulty when asked to apply a concept or line of reasoning to a situation different from that in which it was learned.[3] Explicit guidance is often required for them to do so. In contrast, facility in applying ideas flexibly in diverse contexts is characteristic of physics experts working within their own discipline. The following questions thus arise: Does traditional instruction in advanced topics promote facility in transfer? If not, what specific difficulties with transfer occur?

In the study of relativity beyond the introductory course, students are expected to apply the four-vector in the analysis of problems in kinematics and dynamics.[4] It is reasonable to expect that students will assimilate and gain competence with this new tool by extending their understanding of the already familiar three-vector. The energy-momentum four-vector, an essential tool in the analysis of relativistic collisions, is built from quantities familiar from Newtonian physics: energy (a scalar), and momentum (a three-vector). To secure a meaningful understanding of the conservation of energy-momentum in relativistic collisions, the student must first transfer his or her understanding of energy and momentum to the new context, and then extend this understanding to encompass more complicated situations. Research results described below indicate that despite instruction this transfer often does not occur. In some cases, a transfer of debilitating difficulties prevents meaningful learning.

The first section of the paper identifies specific difficulties that advanced students encounter as they study special relativity. The following section traces these difficulties to difficulties with basic, underlying ideas.

IDENTIFYING DIFFICULTIES WITH ENERGY-MOMENTUM

Two written questions have been used to investigate student understanding of relativistic energy-momentum. These questions – together with overviews of student performance – are described first. Individual student responses are then presented as evidence for specific difficulties.

Non-Interacting Particles Question

In this question, students are shown three systems, each consisting of two non-interacting identical particles. Each particle has mass m and momentum either zero or of magnitude p_o (see Figure 1). Students are told that the moving particles have relativistic speed in the frame shown. They are then asked to rank the systems, first according to the magnitude of total momentum, and then according to mass.

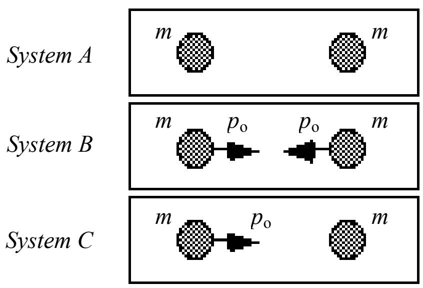

FIGURE 1. Two-particle systems used in the *Non-interacting particles* question.

To rank the systems correctly by momentum, a student can, for each system, sum the corresponding three-momentum components of the individual particles. The resulting system momentum components can be used to determine magnitudes: $P_A = P_B = 0$, and $P_C = p_o$. To rank the systems by mass, the student must first apply $m^2 = E^2 - p^2$ to each particle in order to find its energy in terms of m and p_o.[5] The energies and momenta of the particles can then be summed component-wise to find the energy-momentum four-vector of each system. Finally, the magnitudes of the system four-vectors can be compared to rank the systems by mass: since system B has greater energy but smaller momentum than C, $M_B > M_C$. To compare A and C, a brief calculation must be performed:

$$M_C^2 = [m + (m^2+p_o^2)^{(1/2)}]^2 - p_o^2$$
$$= 2m^2 + 2m(m^2+p_o^2)^{(1/2)} > 4m^2 = M_A^2. \quad (1)$$

The rankings are thus $P_C > P_A = P_B$ and $M_B > M_C > M_A$.

The *Non-interacting particles* question has been administered as a written quiz to 13 students in a junior-level relativity course at the University of Washington. The students had received traditional instruction on energy-momentum. On the momentum task, 8 of 13 answered correctly. The remaining five students all gave the same incorrect ranking: $P_B > P_C > P_A$. Only two students ranked the masses correctly. Six students gave a variety of incorrect rankings while five left the mass ranking blank.

Photon Question

In this variation students were asked to consider systems of two massless photons, rather than two massive particles. Students were told that each photon has energy E. The momentum directions differ from those given in the original problem (see Figure 2).

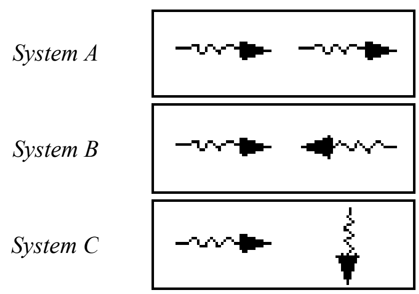

FIGURE 2. Systems used in the *Photon* question.

Students are asked to determine the following in terms of E: (i) the energy-momentum four-vector of each photon, (ii) the energy-momentum four-vector of each system, (iii) the magnitude of the total momentum of each system, and (iv) the mass of each system. To answer correctly, a student can first recognize that for massless particles the magnitude of the momentum is identical to the energy. This allows the energy-momentum four-vector of each photon to be determined. Then reasoning similar to that outlined above can be applied to yield the following results:

TABLE 1. Correct answers to the *Photon* question.

System	Energy-momentum four-vector of system	Magnitude of total momentum of system	Total mass of system
A	(2E, 2E, 0)	2E	0
B	(2E, 0, 0)	0	2E
C	(2E, E, –E)	$E(2)^{1/2}$	$E(2)^{1/2}$

The *Photon* question was administered to 14 students in the same relativity course described above. It was given as a written homework assignment approximately one week after the quiz on which the original question was administered. When determining the system four-vectors, the majority of students responded correctly (12/14 correct for system A, 11/14 for system B. and 10/14 for system C). Correct responses for the momentum and mass, however, were less frequent:

TABLE 2. Number of correct responses on the *Photon* question.

System	$\|\boldsymbol{P}_{system}\|$	$\boldsymbol{M}_{system}$
A	10/14	13/14
B	11/14	7/14
C	7/14	7/14

Discussion Of Specific Difficulties

On both the *Non-interacting particles* question and the *Photon* question, students found determining the invariant mass of the system challenging. Below are described certain types of reasoning that seemed common among the incorrect responses.

Difficulties summing energies and momenta as four-vector components in determining the four-vector of the system. For some students, the failure to determine the system mass correctly seemed related to a failure to treat individual energies and momenta as components of a four-vector. On the *Photon* question one student recognized that $|\boldsymbol{p}| = E$ for an individual photon, but answered (2E, 2E) for the energy-momentum four-vector of system A, (0, 0) for system B, and (E, E) for system C. These answers for the system energy, correct for system A but incorrect for B and C, suggest a tendency to combine energies inappropriately as signed quantities. Furthermore, the incorrect momentum components given for system C indicate a failure to recognize that the total three-momentum vector must include at least two non-zero components. It is as if the student has applied a simplistic rule of "complete reinforcement" for photons propagating in the same direction, "complete cancellation" for opposite directions, and "non-combination" for orthogonal directions.

The incorrect energy-momentum four-vectors led the student to the incorrect answer of zero total mass for each system. In all, 4 out of the 14 student responses indicated that the masses of all three systems are the same.

Tendency to confuse mass and momentum. One student gave correct energy-momentum four-vectors for each photon system A-C. He went on, however, to answer incorrectly for the total momentum: "$P_A^2 = (2E)^2–(2E)^2 = 0$; $P_B^2 = (2E)^2$; $P_C^2 = [(2E)^2–E^2–E^2]$." The student confuses the invariant system mass for the total three-momentum. Another student, who like the first obtained correct energy-momentum four-vectors for each system, gave a slightly different incorrect response for momentum: "$P_A^2 = P_x^2–0^2–0^2$; $P_B^2 = 0^2–0^2–0^2$; $P_C^2 = P_x^2–P_y^2–0^2$." This response does not incorporate the energy part of the four-vector, as the previous response did, but inappropriately applies the algorithm for computing the magnitude of a four-vector to determine the magnitude of three-momentum.

TRACING DIFFICULTIES WITH ENERGY-MOMENTUM TO DIFFICULTIES WITH THREE-MOMENTUM

The inability of some students to determine the invariant system mass seemed related to difficulties with concepts and reasoning not specific to the relativistic context. On the *Non-interacting particles* question, for example, 5 of 13 students gave the incorrect system momentum ranking $P_B > P_C > P_A$. One of these students reasoned that "the magnitude of $P_B = 2p_o$, the magnitude of $P_C = p_o$, and the magnitude of $P_A = 0$." Another explained "B > C > A; the total momentum is $p_1 + p_2$ and has nothing to do with direction."

This tendency to sum vector magnitudes when finding the momentum of a system – a serious difficulty with a foundational idea – prompts consideration of the following possibilities: 1) students capable of correctly applying the momentum concepts in non-relativistic contexts are failing to transfer their understandings to the new, more complex situation, and/or 2) difficulties with momentum can be traced back to non-relativistic contexts and are not fully resolved through advanced study. We thus must ask: To what extent do students experience related difficulties with momentum in non-relativistic contexts?

To address this, an additional written question was administered. Students were told that two identical objects – with momenta of the same magnitude –

collide and stick together to form a new, composite object. Students were then asked to compare p_o, the magnitude of the momentum of the original objects, with P_C, the magnitude of the momentum of the composite object.[5] To answer correctly, a student can add the momentum vectors of the two original objects to determine that of the composite. This leads to the comparison $p_o < P_C < 2p_o$.

The question has been administered to 304 students in two sections of the introductory calculus-based mechanics course at the University of Washington. Students had received traditional instruction on conservation of momentum, including quantitative homework problems on collisions in two dimensions. Performance in the two sections was comparable: about 30% gave the correct response, while the most common incorrect response, $P_C = 2p_o$, was given by about 35%. The following explanations are typical of students who gave the most common incorrect response:

> *"Magnitude doesn't consider direction. Since both particles have equal mass and magnitude of momentum, $P_C = 2p_o$."*

The student explicitly considers direction, but indicates that it does not have bearing on the magnitude of the total momentum. This type of incorrect reasoning is similar to that used by more advanced students in the relativistic context of the *Photon* question. The difficulties with momentum that prevented advanced students from answering correctly about invariant mass can be traced to similar difficulties in a Newtonian context.

A variation of the question has been administered to 53 physics graduate students and advanced undergraduates in the teaching seminar at the University of Washington. Although about 75% answered correctly, 9 out of the 53 students gave incorrect responses similar to the $P_C = 2p_o$ response given by introductory students on the original version of the question. This suggests that the tendency to add magnitudes when summing momentum vectors is not fully resolved through advanced study. Students may be transferring this unresolved difficulty to situations involving relativistic energy-momentum as they progress to more advanced material.

Another domain in which the ability of advanced students to transfer knowledge has been examined is relativistic electrodynamics. In a study involving over 100 students in two sections of the honors introductory course at the University of Washington, student ability to relate current as well as electric and magnetic forces in multiple frames was probed. More than half of these students failed to recognize that the current in a wire maintains the same value under a Galilean transformation. Many students seemed unable to apply the operational definition of current in a frame moving relative to the laboratory.

CONCLUSION

As physics students engage with the study of advanced material, they must bring basic ideas to bear in new, more complex situations. This requires first a solid foundation of understanding of the underlying ideas, and then a vertical transfer to the more complicated situation. The examples presented here suggest that in some cases students instead transfer a deficient understanding of the underlying concepts. In order to promote meaningful learning of advanced topics, difficulties with basic ideas must be addressed.

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NOTES

1. Barnett, S.M., and Ceci, S.J., "When and where do we apply what we learn? A taxonomy for transfer" in *Psychological-Bulletin* **128 (4)**, 612-637 (2002).
2. Bransford, J.D., and Schwartz, D., "Rethinking transfer: A simple proposal with multiple implications," in *Review of Research in Education,* ed. A. Iran-Nejad and P.D. Pearson, Washington, D.C.: AERA, 1999, pp. 61-100.
3. See, for example, Arons, A.B., *A Guide to Introductory Physics Teaching*, New York: Wiley and Sons, 1990.
4. For a lucid presentation of the ideas at the sophomore level, see Taylor, E.F. and Wheeler, J.A., *Spacetime Physics*, New York: W.H. Freeman, 1992.
5. Mass is identified with the magnitude of the energy-momentum four-vector; units in which c=1 are used.
6. For more details, see Boudreaux, A., "An Investigation of Student Understanding of Galilean Relativity," Ph.D. dissertation, Department of Physics, University of Washington, 2002 (unpublished).