Same to Us, Different to Them: Numeric Computation versus Symbolic Representation

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Abstract. Data from nearly 900 students was used to measure differences in performance on numeric and symbolic questions. Symbolic versions of two numeric kinematics questions were created by replacing numeric values with symbolic variables. The mean score on one of the numeric questions was 50% higher than the analogous symbolic question. An analysis of the written work revealed that the primary identifiable error when working on the symbolic problems was a confusion of the meaning of the variables. The paper concludes with a discussion of possible theoretical explanations and plans for future follow-up studies.

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INTRODUCTION

Algebra is an essential skill for students in physics. For some students the lack of algebraic facility acts as a barrier to success in physics. Studies from the mathematics education research community have shown that many students learning algebra have difficulties understanding the meaning of symbolic variables and dealing with unevaluated expressions [1]. We performed a study to measure the effect of such algebraic difficulties in the context of a physics exam.

We modified exam questions to see the effect of numeric computational and symbolic representational cues on student performance. We hypothesized that changing the numeric quantities in the problem to symbolic variables would change the way students thought about and approached the solution to the problems. Specifically, we thought that students would have difficulties with problems that required them to plug an unevaluated symbolic expression into another symbolic equation.

METHODS

The subjects of the study were students who were enrolled in the calculus-based introductory mechanics course, Physics 211, at the University of Illinois, Urbana-Champaign in the spring 2006 semester. Students in Physics 211 take three multiple-choice midterm exams and a cumulative final. While numeric questions are the most common type of question, symbolic questions are not uncommon. There were 894 students who completed one of the two randomly administered versions of the final exam.

We placed a pair of kinematics questions dealing with cars on each version of the final exam. On final exam 1, numeric values were used and on final exam 2, symbolic variables were used. The questions are shown in Figure 1.

Minor modifications were made to discourage cheating. The order of the problems and the order of the choices were reversed from one final to the other. The same choices are present in both versions of each problem, except final 1 question 7 and its partner final 2 question 6 where two choices do not agree. Both choices, however, were selected by less than one percent of the students.

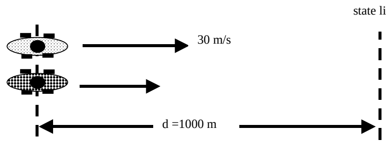
We also placed a pair of Newton's 2^{nd} law problems on each version of the final exam whose solutions involved intermediate variables that cancelled out of the final equation. The version on final 1 supplied numeric values for the intermediate variables and the version on final 2 supplied symbols for the intermediate variables. The analysis for these questions is still ongoing and will not be discussed in this paper.

RESULTS

As we expected, the mean score of the questions that used symbols were lower than the questions that did not. Table 1 lists the mean and the standard error for each question.

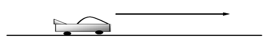
Final 1

A bank robber is racing towards the state line at a constant speed of 30 m/s and passes a cop who is parked 1000 m from the state line. At the moment the robber passes him, the cop accelerates from rest with a constant acceleration to try to catch the robber before he reaches the state line.



6. What is the minimum acceleration that the cop must have in order to catch the robber before he reaches the state line? (Ignore any reaction time.)

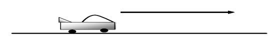
a. 0.23 m/s^2 (< 1%) b. 0.45 m/s^2 (7%) c. 0.90 m/s^2 (6%)
d. 1.80 m/s^2 (85%) e. 3.60 m/s^2 (< 1%)
(More than 1 choice: 2%)



7. A car can go from 0 to 60 m/s in 8 seconds. At what distance, d, from the start (at rest) is the car traveling 30 m/s? (Assume a constant acceleration.)

a. 30 m (< 1%) **b. 60 m (94%)**
c. 90 m (<1%)(No corresponding choice)
d. 120 m (2%) e. 240 m (2%)
(More than 1 choice: 0%)

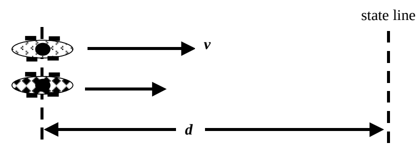
Final 2



6. A car can go from 0 to a speed v_1 in t seconds. At what distance, d, from the start (at rest) is the car traveling (v_1 / 2)? (Assume constant acceleration.)

a. $d = v_1 t$ (< 1%) (No corresponding choice)
b. $d = \frac{v_1 t}{2}$ (25%) c. $d = \frac{v_1 t}{4}$ (27%)
d. $d = \frac{v_1 t}{8}$ **(45%)** e. $d = \frac{v_1 t}{16}$ (< 1%)
(More than 1 choice: 3%)

7. A bank robber is racing towards the state line at a constant speed of ***v*** and passes a cop who is parked ***d*** from the state line. At the moment the robber passes him, the cop accelerates from rest with a constant acceleration to try to catch the robber before he reaches the state line.



What is the minimum acceleration that the cop must have in order to catch the robber before he reaches the state line? (ignore any reaction time).

a. $a_{min} = (2v)^2/d$ (3%) **b. $a_{min} = 2v^2/d$ (57%)**
c. $a_{min} = v^2/d$ (6%) d. $a_{min} = v^2/(2d)$ (29%)
e. $a_{min} = v^2/(4d)$ (< 1%)
(More than 1 choice: 4%)

FIGURE 1. Questions placed on the final exam. Popularity of each choice is shown in parentheses and the correct choice is in bold.

TABLE 1. Exam Question Mean and Standard Error

	Final 1 (Numeric, N=453)	**Final 2 (Symbolic, N=441)**
"Bank robber"	85.0% ± 1.7%	57.4% ± 2.4%
"A car can go"	94.5% ± 1.1%	44.7% ± 2.4%

To test the equivalence of the groups we compared the mean midterm exam grade for each group. The mean midterm exam grade and standard error for the students who were given final exam 1 was 72.2% ± 0.6%; for the students who were given final exam 2 the mean midterm exam grade and standard error was 71.9% ± 0.6%. According to this measure the two groups are virtually indistinguishable.

Analysis of Written Work

At the conclusion of the exam, all students were required to turn in all exam materials. The exam packets were collected to keep the exam secure for future use and to resolve student concerns, but were not graded. Students, therefore, were not required to show any work in these packets.

We coded a sample of student written work (89 students). Table 2 demonstrates that the mean score for each of the questions for our sample was consistent with that of the overall population. The coding scheme developed and the numbers of students making each error are shown in Table 3 and Table 4.

TABLE 2. Exam Question Mean and Standard Error for the Sample of 89 Students.

	Final 1 (Numeric, N=43)	**Final 2 (Symbolic, N=46)**
"Bank robber"	83.7% ± 5.6%	60.9% ± 7.2%
"A car can go"	93.0% ± 3.9%	37.0 ± 7.1%

Even though the students were not graded based on their written solutions; a great majority of the exams from our sample showed work. In most cases it was clear what the student was doing, but there were some

TABLE 3. Coding of student work for the "A car can go" problem

Final 2 Question 6 (A car can go) Codes	Number of Students
Correct	17
Using $v_1/2$ = at and $(v_1/2)^2$ = 2ad. Here t is misunderstood to mean the time it takes for the car to reach a speed $v_1/2$. They get the incorrect result d = v_1t/4.	4
Using v_1= at and d = (1/2)at^2. Here t in the second equation in misinterpreted and they are actually finding the distance when the car reaches a speed v_1 rather than the distance when it reaches a speed $v_1/2$. They get the incorrect result d = v_1t/2.	1
Using $v_1/2$ = at and d = (1/2)at^2. Here t in both equations is misinterpreted. See errors 1 and 2. They get the incorrect result d = v_1t/4.	5
Use (v_1/2)(t) = d or use (v_1/2)(t/2) = d. They use the equations for constant velocity. In the first they also misinterpret the meaning of the variable t. They get the choices d = v_1t / 2 and d = v_1t / 4 respectively.	2
An algebra error	3
Indeterminate	9
No work	3
Other	2

TABLE 4. Coding of student work for the "Bank robber" problem

Final 2 Question 7 (Bank robber) Codes	Number of Students
Correct	28
Using v^2 = 2ad. Here the v is used as if it were the cop's final velocity when he reaches the bank robber, but in the statement of the problem it is actually the bank robber's speed. They get the choice a = v^2/(2d).	11
Using t = d/v and v = at. The v in the second equation is used as if the cop's final velocity is the velocity of the bank robber. They get the choice a = v^2/d.	2
An algebra error	1
Indeterminate	2
No work	1
Other	1

cases in which the student's method was indeterminate. We also found that the main difficulty was not related to their inability to combine symbolic equations. The main difficulty seemed to be that when working symbolically the students often confused the meaning of the variables. They would confuse the meaning of the variable "t" in the "A car can go" problem and the meaning of the variable "v" in the "bank robber" problem.

DISCUSSION

Even though no one student received both the numeric and symbolic version of a particular question, the data suggests that 30% to 50% of the students would be able to solve a numeric version correctly but not an analogous symbolic problem. Even though we hypothesized that the students would have difficulty combining multiple symbolic equations, no evidence was found to support this hypothesis. Instead many students successfully combined symbolic equations, but did so with variable confusions.

The fact that the mean score on the 2nd question was higher than the 1st question on each final could be an indication of an effect due to the order of the questions. But because we find that the mean score on the symbolic versions are the lower than the numeric versions across exams, the order effect is small compared to the numeric/symbolic effect.

From an expert perspective the solutions to the numeric and symbolic questions are identical. The types of errors made, however, make it clear that the questions are not identical to the students. In the sample of written work we analyzed there were more students who made variable confusion errors on the symbolic questions than the total number of students who incorrectly answered the numeric questions.

Written Work and Theoretical Descriptions

The analysis of the students' written work on the symbolic version of the "bank robber" problem showed that 11/18 of the students who marked an incorrect choice also used the equation $v^2=2ad$. This error appears to be a matching of the variable "v" given in the problem with the "v_f" in the general equation "$v_f^2 = v_o^2 + 2a\Delta x$" on the equation sheet. When they make this error they are confusing the velocity of the bank robber with the final velocity of the police officer. This may be especially attractive to students because this solution only requires a single equation in order to obtain one of the choices while the correct solution requires two equations coupled by the time.

The resources framework proposed by David Hammer and colleagues [2] seems to describe certain aspects of our data. The resources framework posits knowledge pieces whose activation depends on the way the situation/task is framed. The framing determines what resources are activated and the types of activities (games) that are appropriate. The differences we observed between numeric and symbolic problems may be described as a difference in the way many students frame the two types of problems. Especially in the symbolic "bank robber" problem it appears that many students may have framed the problem as a "symbol matching" activity. It appears, on the other hand, that similar students framed the numeric version of the same problem as an activity involving the use of general equations and numeric quantities in a meaningful way.

One might imagine students framing the symbolic problems as a conceptual exercise that could be solved without the necessity of combining equations. But because the majority of the exams showed attempts to combine equations, we do not believe that this interpretation is a major factor in the results we found.

Similar theoretical frameworks have been described in the mathematics education research literature. Sfard [3] posited the existence of a duality of the procedural and the structural perspectives of mathematical concepts. Symbolic representations exemplify the dual nature of mathematical concepts. Symbolic equations can be thought of procedurally as a computation, as well as structurally as a statement of a relationship. Sfard's framework differs from the resources framework in that during the development of a mathematical concept the operational conception must precede the structural conception. In the resources framework no hierarchy of resources exists.

The analysis of the students' written work on the "A car can go" problem is less clear. The main identifiable error can by seen by combining the 2nd and 4th rows of the Table 3. 9/29 students who chose an incorrect choice used the equation "$(v_1/2)=at$" to determine the acceleration of the car. The general equation, "$v_f = v_o + at$" is appropriate, but they confuse the variable "t" in the equation with time to reach the velocity "$(v_1/2)$". To form this incorrect equation they combined information from two different sentences in the problem. When deciding what to substitute for "v_f" in the general equation only "v_1" and "$(v_1/2)$" have the correct units for the replacement. They may be influenced to choose "$(v_1/2)$" because that is the quantity of focus in the question.

Unlike the "bank robber" problem, in the "A car can go" analysis of the students' written work we found that 9/29 of the students who made the incorrect choice showed work that was indeterminate. These students either showed partial equations, many conflicting equations, or only a few sparse equations.

Cognitive load theory [4] can also be used to describe the data. This theory focuses on how cognitive resources are used. The cognitive demands on novices are much higher than on experts because novices must frequently store information in their working memory which can become quickly overloaded, while an expert can rely on pieces of compiled knowledge to minimize the strain on their working memory. It may be that many students have great difficulty simultaneously attending to the meaning of variables, the meaning of the symbolic expression, and the symbolic manipulations. The common incorrect equation "$(v_1/2)=at$" from the symbolic "A car can go" problem may be a result of trying to coordinate different pieces of information from different sentences along with the meaning of the equation on the equation sheet. This theory also seems to explain why there are so many students whose written work is indeterminate. Students whose working memory is overloaded would be expected to produce multiple and contradictory equations in the process of coordinating different information.

FUTURE WORK

In future experiments we would like to further study the differences between numeric and symbolic problems. In one proposed experiment, we plan on replacing the symbols used with uncommon symbols (e.g. b, c, q) to see if by increasing the barrier for symbol matching students would be more likely to treat these uncommon symbols in a way similar to numeric quantities. In another proposed experiment, we plan on replacing the confused variable with a numeric value, while keeping the other variables symbolic. In this variation the students would still have to setup an equation and would still have to carry an unevaluated expression from one equation to the next.

These experiments would help us identify what factors influence how students frame numeric and symbolic problems. These experiments would also allow us to determine the effect of varying the amount of symbolic information that the students have to coordinate, and thus the effects of cognitive load.

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