Students' Understanding of Dot Product as a Projection in No-context, Work and Electric Flux Problems

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Abstract. In this article we investigate students' understanding of dot product as a projection. In the first part, we compare students' performance in three isomorphic multiple-choice problems: no-context, work and electric flux. We administered one of the three problems to 422 students who were in the process of completing required introductory physics courses. In the second part, we analyze the students' ability to connect the physical concepts with the dot product's formal representation. We carried out interviews with 14 students, in which they were asked to solve the same three isomorphic problems. Following the tests, we found a difference that was statistically significant: both physical context problems helped students select the projection interpretation option. However, the percentages of students that selected this option remained very low in the three problems. Moreover, during the interviews we noticed that students had serious difficulties in developing a coherent conceptual framework between the physical concepts and the dot product's formal representation.

Keywords: dot product, projection, work, electric flux, misconceptions.
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INTRODUCTION

Physics education researchers have established a research-based generalization on student learning that indicates that students often fail to make connections between concepts and formal representation, even following instruction [1]. In this article we investigate students' difficulties with the formal representation of dot product projection.

Some researchers have studied students' difficulties with dot product´s calculation [2-5] and geometric interpretation [4, 5]. However, studies that directly focus on the understanding of dot product as a projection have not yet been done. Van Deventer [4] designed a multiple-choice problem that specifically evaluated its scalar nature. In our research [5], we administered an open-ended problem that didn't allow us to deduce if students comprehended the projection interpretation.

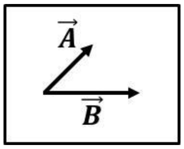
The three objectives of this study are: 1) To analyze students' difficulties with the dot product projection's formal representation in no-context problems, 2) to compare students' performances in each of the three isomorphic problems (no-context, work and electric flux) that evaluate the projection's formal representation, and 3) to analyze students' difficulties in connecting the physical concepts (work and flux) with the dot product projection's formal representation.

METHODOLOGY

This study was conducted at a large private Mexican university and was based on the testing of three isomorphic problems. Figure 1 shows the statements of the three problems and only the multiple-choice options for the no-context problem. Note that the options for the context problems have the same structure. These distractors were constructed from the results of our previous research based on open-ended problem analysis [5].

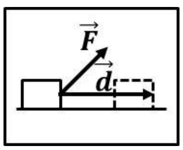
This study was completed with data from two consecutive semesters. To address the first two objectives, during the first semester we administered a test with one of the isomorphic problems to 422 students in a calculus-based Electricity and Magnetism course (the last of three introductory physics courses that students take in this institution). We divided students into three different groups of approximately 140 students each, following the methodology used previously [6]. Each randomly-selected group answered only one problem.

To address the third objective, during the second semester we carried out individual interviews with 14 self-selected students from the same course, using a think-aloud protocol [7]. Students answered the three same isomorphic problems (Fig. 1), without including the dot product definition in the two context questions. The sequence included: 1) the no-context problem, 2) a unit-vector problem (not shown), 3) a work context problem, 4) a scalar multiplication of a vector problem (not shown), and 5) the flux problem. Problems 2 & 4 were incorporated to prevent the students from directly relating the isomorphic problems to each other. Interviews and testing were conducted in Spanish.

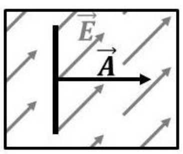


No context problem. In the figure there are two vectors **A** and **B**. Which option is the best interpretation of the dot product (**A**·**B**)?

(a) The magnitude of a vector between **A** and **B** pointing up to the right.
(b) A vector in the direction of **B**.
(c) A vector in the direction of **A**.
(d) A vector between **A** and **B** pointing up to the right.
(e) The projection of vector **A** onto vector **B** multiplied by the magnitude of vector **B**.
(f) A vector perpendicular to both vectors.



Work problem. The figure shows a force **F** exerted on a box. The box moves a displacement **d**. Which option is the best interpretation of the work done by the force **F**, *defining this work as the dot product (**F**·**d**)*?



Electric flux problem. The figure shows a side view of a plane surface in a region with a uniform electric field **E**. The vector area **A** is perpendicular to the surface. Which option is the best interpretation of the electric flux through the surface, *defining this flux as the dot product (**E**·**A**)*?

FIGURE 1. Isomorphic problems used during the testing and interviews. In the context problems of the test, we included the dot product definition (in italics), but in the interviews we did not.

TEST RESULTS

This section is divided into two subsections addressing the first two objectives of this study.

Difficulties in the No-Context Problem

In Table 1 (no-context problem), we noticed that a low percentage of students (21%) selected the correct option (scalar projection, option e). This result is surprising, since these students had already used dot product in their mechanics course (work) and also in their electricity and magnetism course (electric flux and potential) before the test.

In Table 1, the results of two incorrect answers (option a: 36%; option d: 28%) are higher than that of the correct answer. These two options refer to a vector between the two vectors. Option d) refers to the vector and option a) to the magnitude of this vector. We categorized the students' reasoning to understand why they chose these incorrect options. We found that the most common incorrect reasoning by students that chose option a) was to relate the scalar nature of the dot product with the magnitude of a vector. An example of this incorrect reasoning is: "*Dot product is a scalar, that´s why its result is a magnitude and not a vector*". On the other hand, we detected, as Van Deventer [4], that the most common incorrect reasoning by students who chose option d) was to relate the dot product with the addition of vectors.

Table 1 also shows that 9% of students interpreted the dot product as a vector in the direction of vector **B** (option b). We found that the most common incorrect reasoning was based on an improper calculation of the dot product using unit vector notation. One student´s answer that illustrates this incorrect reasoning is: "*To get the dot product, one multiplies* $A{\cdot}B = (x\hat{\imath} + x\hat{\jmath}){\cdot}(y\hat{\imath}) = xy\hat{\imath}$. *As can be seen,* $i{\cdot}\hat{\jmath}$ *cancels out and only the multiplication* $\hat{\imath}{\cdot}\hat{\imath}$ *remains*". (Note also that these students may think of dot product as a vector projection). Additionally, Table 1 shows that 6% of students interpreted the dot product as a perpendicular vector (option f), which is actually the cross product interpretation. We found that the most common incorrect reasoning was based on an improper calculation of dot product using unit vector notation. An example of a student´s answer showing this incorrect reasoning is: *"Dot product* $\hat{\imath}{\cdot}\hat{\jmath}$ *results in k direction"*. Note finally, that the majority of students chose a scalar option (options a and e) in this problem, which suggests that most students recognize that the dot product should be a scalar, not a vector.

Comparing Students' Performances

To compare students' performances we used Fisher´s exact test to determine whether the differences were statistically significant. This test was used instead of a chi square test to avoid issues with low cell counts [8]. Table 1 shows the percentages of students who chose each multiple-choice option in the three problems, and the arrows indicate which pair of options was significantly different from each other ($p < .05$). The first interesting result is that we found significant differences between the no-context and both contexts problems. These differences show how sensitive students' answers are to the contexts of the problems. In contrast, we didn't find differences between both contexts problems, showing that the differences between the contexts do not affect significantly students' answers.

TABLE 1. Percentages of students who chose each option in the three problems. The correct answer for the three problems is option e). Note that each student answered only one problem. The arrows indicate which options have significantly* different distributions between two problems. *Statistically significant, determined by $p < .05$ on Fisher's exact test.

Responses	No-context	Work	Electric flux
(a) The magnitude of a vector between the two vectors	36%	22%	20%
(b) A vector pointing at a 0° angle	9%	11%	6%
(c) A vector pointing at a 45° angle.	0%	2%	5%
(d) A vector between the two vectors	28%	23%	29%
(e) The projection of one vector onto a second vector multiplied by the magnitude of the second vector	21%	39%	32%
(f) A vector perpendicular to both vectors	6%	3%	7%

We found three significant differences between the no-context and both contexts problems: 1) We found a difference in the selection of option e (correct answer) between the no-context and both contexts problems, 2) we detected a significant difference in the selection of option a (incorrect option that refers to the magnitude of a vector) between the no-context and both contexts problems, and 3) we found a significant difference in the selection of option c (incorrect option that refers to a vector at 45°) only between the no-context and the electric flux problems. Note that option c refers to a vector in the direction of the electric field **E** in the electric flux problem. Next, we analyze these three differences in the students' reasoning.

The first difference found was in the selection of the correct option. Both contexts helped students selecting this option. Analyzing the students' reasoning, we found some evidence that the understanding of the physical concepts helped students construct the projection interpretation. An example of a student's reasoning for selecting the correct option in the work problem is: "*The box moves to the right, so it is logical to think that the force that causes this movement is the x-component of the force vector. Therefore it is the projection of F onto d.*" It is interesting to note that we can't state from this type of reasoning whether this student knew the dot product interpretation as a projection. What we can state is that understanding the work concept itself helped him select the correct interpretation. The connection between the physical concepts and dot product projection's formal representation will be analyzed in the interview results. Finally, note that although the physical contexts help students select the projection interpretation, the performance remains low (< 40%).

The second difference found is in the selection of the incorrect option a. The no-context problem triggers this selection. As mentioned before, the most common reasoning used in selecting this option in the no-context problem is to relate the dot products' scalar nature with the magnitude of a vector. We also found in both contexts problems that the most common reasoning was either to relate the scalar nature of work and flux or the scalar nature of the dot product to the magnitude of a vector.

The third difference found is in the selection of option c between the no-context and the flux problems. The latter triggers the selection of this option. Analyzing the students' reasoning, we found that the majority had a misconception in the understanding of the electric flux concept (not in the dot product) and believed that the flux is a vector that "*runs in the same direction as the electric field.*"

INTERVIEW RESULTS

Difficulties in Connecting the Physical Concepts with the Formal Representation

In this section we address the third objective of the study. Table 2 shows the responses of the 14 interviewed students. Notice that the students followed different patterns of answers among each other. This was the first evidence of students' difficulties.

TABLE 2. Responses of the 14 interviewed students. (The given answer and the option chosen are specified).

St.	No-context	Work	Electric flux
1	Magnitude a)	Vector Bet. d)	Vector Bet. d)
2	Magnitude a)	Projection e)	Projection e)
3	Projection e)	Perp. Vector f)	Magnitude a)
4	Magnitude a)	Vector 0° b)	Vector 45° c)
5	Perp. Vector f)	Vector 0° b)	Perp. Vector f)
6	Vector Bet. d)	Projection e)	Projection e)
7	Vector Bet. d)	Vector Bet. d)	Magnitude a)
8	Magnitude a)	Magnitude a)	Magnitude a)
9	Magnitude a)	Vector 0° b)	Perp. Vector f)
10	Vector Bet. d)	Vector 0° b)	Vector 0° b)
11	Vector Bet. d)	Vector Bet. d)	Vector Bet. d)
12	Vector 0° b)	Vector 0° b)	Vector 0° b)
13	Projection e)	Projection e)	Projection e)
14	Vector Bet. d)	Vector Bet. d)	Perp. Vector f)

The data of Table 2 confirm the previous conclusion, that only a small percentage of students (4 of 14) chose the projection option in any of the three problems (Students 2, 3, 6 & 13). The interesting result is that not all of this small percentage of students selected the projection interpretation in the three problems. In fact, only one of them did so (student 13). If we focus only in the students' answers we can notice that student 13 doesn't exhibit a specific difficulty connecting the physical concepts with the dot product formal representation, but students 2, 3 and 6 do exhibit specific difficulties. Next we analyze the lines of reasoning of these three students to exemplify these difficulties.

Students 2 and 6 select the correct interpretation in both context problems, but an incorrect one in the no-context problem. Student 2 chooses the magnitude of a vector option based on the fact that the dot product is a scalar, and Student 6 selects the vector between the two vectors option, relating the dot product with an addition of vector. Both students justify their correct selection in the context problems based on an understanding of the physical concepts (with reasoning such as that presented previously). By contrast, student 3 selects the projection option in the no-context problem, but not in the context problems. This student recalls from an introductory course that "dot product is a projection", but when questioned further, states that he doesn't "remember" how to justify this fact. He then selects the perpendicular vector interpretation in the work problem, because he thinks that the work is defined by the cross product, and selects the magnitude option in the flux problem, based on the fact that the flux is a scalar. This analysis illustrates the type of difficulties that students (who selected the projection interpretation in any of the isomorphic problems) have in developing a coherent framework. (Note that by "coherent framework" we mean a framework in which students connect the physical concepts to the dot product formal representation).

We can also analyze the sequence of answers of the 10 students (from the 14) that didn't choose the projection interpretation in any problems. We found that: 1) three of them (students 8, 11 & 12) selected the same representation in the three problems, showing that they connected the three problems in some way, 2) two of them (students 7 & 14) connected only the no-context and the work problems, 3) two of them (students 1 & 10) connected only both context problems, 4) one of them (student 5) connected only the no-context and the flux problem, and two of them (students 4 & 9) didn't connect any of the problems. It's interesting that only three out of ten students (independently of the lines of reasoning that they follow) selected the same representation in the three problems. This fact shows that students that have difficulties with the projection interpretation also have difficulties in developing a coherent framework.

CONCLUSIONS

Physics education has established a generalization on student learning which states that students often fail to make connections between concepts and formal representation even after instruction [1]. In this article we demonstrate that this failure to make connections is very serious with regard to dot product projection's formal representation. As a result, we found that student performance in the no-context problem was very low (21%), and that although the physical contexts help students to select the projection interpretation, the performances remained low (< 40%). Physics education has also established that students' acquired knowledge is often quite incoherent and tends to be fragmented [9]. In the interviews, we proved that the incoherence in students' knowledge of dot product projection's formal representation is considerable and that students have serious difficulties in connecting the physical concepts with this representation. These results present evidence of the need for designing strategies that promote: 1) the understanding of dot product projection's formal representation, and 2) the development of a coherent conceptual framework between the physical concepts and this representation. We also suggest that these strategies take into account the misconceptions and frequent incorrect reasoning highlighted in this article.

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