

# When Basic Changes To A Solution Suggest Meaningful Differences In Mathematics

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**Abstract.** When solving two integrals arising from the separation of variables in a first order linear differential equation, students have multiple correct choices for how to proceed. They might set limits on both integrals or use integration constants on both or only one equation. In each case, the physical meaning of the mathematics is equivalent. But, how students choose to represent the mathematics can tell us much about what they are thinking. We observe students debating how to integrate the quantity  $dt$ . One student seeks a general function that works for everyone, and does not wish to specify the value of the integration constant. Another student seeks a function consistent with the specific physics problem. They compromise by using a constant, undefined in value for one student, zero in value for the other.

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## INTRODUCTION

Students solving physics problems often have multiple mathematical paths with which to correctly approach the problem. How they choose their solutions can depend not on the correctness of a given solution method but on other factors, such as the form of the equation which they expect to find, or the details of what they want the solution to describe. We present such a situation in the context of integration, where one can use either an integration constant or integration limits to correctly arrive at a solution.

We have previously reported on the debate held by two students, Max and Phil, trying to integrate the value of  $dt$  in a separable first-order differential equation [1]. In this paper, we revisit the same data set and analyze it from a new perspective. Rather than looking at a schema theory analysis (e.g., procedural resources) used by the two students, we look instead at how the two students seek to find agreement in a situation where their goals about the problem solution differ strongly. We find that Max connects the physical meaning of the problem to the mathematical formalism, and sees constants as having values associated with a meaningful (yet arbitrary) interpretation of the system. We find that Phil seeks generality, and sees integration constants more as place-holders which should not be specified too closely, otherwise the function cannot be used by many people.

## DEBATING THE PHYSICS PROBLEM

Students working on a group quiz were asked to find the motion of a thrown object, subject to only a quadratic air resistance force. One would expect such an object to slow down to a stop, after initially starting with some velocity. To solve this problem, one can separate variables and integrate each side independently. As part of this activity, two students had the following debate while trying to integrate the simple quantity,  $dt$ . We leave out other information about the problem they were solving, namely the integration of the  $dv$  side of the equation. We do point out that students were not given explicit initial conditions but had to infer them from the problem statement.

We present the full dialogue (with one gap) that Phil and Max (aliases) engaged in. A third student, Daryl, was part of this group but did not speak during this time. (Throughout, we write  $t_0$  when students say "t-nought.")

Phil: [this] equals  $t$  plus  $c$ .

Max: No, this is where you should be going just plain  $t$ , or  $t_0$  to  $t$ , or zero to  $t$ , because initial time is when you throw it to some time, so just go to  $t$  instead of  $t$  plus  $c$ . <long silence> Because then your integration of time would probably be going from zero, when you throw it, to  $t$ , some time later. So you don't need  $c$  there.

Phil: Hmmmm... or you could do  $t$  plus  $t_0$ . You need a constant though. I mean yeah, if you did it

Max: Why would you need a constant in time? You just go from 0 to  $t$ .

Phil: Because it's <sigh> –

Max: –  $t_0$  to  $t$  –

Phil: – that's not a function, welllll

Max: well, you could go from  $t_0$  to  $t$ , but  $t_0$  is just initially zero when you start throwing the ball.

Phil: If you, ssssss – no, you can't do that, because then you're not setting up a function. You're setting up a function that's only OK in certain cases, you see what I mean?

Shortly after (leaving out a few off topic statements by the two), Phil says: I just can't imagine integrating without having an extra constant. I guess if you said  $t_0$  is zero, so, I mean, yeah –

Max: – but  $t_0$  is the initial time

Phil: But you don't know that's zero.

Max: You can just mark it as zero, anyway <short pause> for consistency. Why, why do you need some extra time there?

Phil: Yeah, but then you're not creating an objective function that anyone can use. You're making a function that you know, and that's great, but it's not really, I mean,

Max: Well, no.

Phil: the math has to cover everything. We'll compromise, we'll say  $t$  minus  $t_0$ . Write that. That work?

Max: That's fine.

Phil: OK.

In the following sections we interpret first Phil and then Max and how they discussed their ideas. We follow with a discussion of the meaning of variable for each student. Finally, we discuss the implications as they relate to issues of listening to our students and how our students listen to each other.

## A GENERALIZABLE FUNCTION THAT WORKS FOR ANYONE

Phil starts off by asserting that the solution to the problem is " $t + C$ ," and ends it saying " $t - t_0$ ." There are two changes in his response. First, he changes the name of the constant from  $C$  to  $t_0$ . Second, he changes the sign of the function from plus to minus. Both these changes suggest something about his view of the mathematics.

When Phil presents the solution, it is likely that he is only thinking of the typical, memorized solution to the integral of  $dt$ . We assume he is comfortable with the mathematics (a conclusion strongly supported by other observations of Phil during a full semester of instruction). His explanation for wanting a constant, "I just can't imagine integrating without having an extra constant," suggests that he is following a rote procedure. He does not explain his reasons for using an integration constant until he responds to Max.

In explaining his function and the need for an unspecified constant, Phil says he seeks "an objective function that anyone can use," not one that is "only okay in certain cases." He does not assign value to the integration con-

stant and does not accept when Max wishes to. He does change its name, though. Moving from  $C$  to  $t_0$  suggests that Max's terminology influences him.

Though it might seem to come from out of the blue at the end of the problem, Phil's change from a plus in  $t + C$  to a minus in  $t - t_0$  can be seen as a slow development in response to Max. First, Phil suggests moving from  $t + C$  to  $t + t_0$ . He tests out the idea of having  $t_0$  be zero, and seems to reject it. He shifts from talking about adding the  $t_0$  to "having an integration constant," a non-mathematical statement. One could think of this as a change from a process of adding to a state of having. This metaphor of possession suggests that an integration constant is required, far more than adding the constant suggests so. Finally, Phil suggests subtraction of the constant. This is the first time subtraction is discussed. As shown below, Max has been arguing as if he has been using integration limits in solving the problem, rather than an integration constant. So, it seems that the compromise that Phil proposes is to use the terminology and form of integration limits (the high bound of an unspecified, running variable, the low bound with the initial time) while keeping the constant that he insists upon. He shifts from talking about adding a constant to subtracting  $t_0$ . This suggests something about how he views the quantity  $t_0$ , as discussed further below.

## A SPECIFIC FUNCTION CONSISTENT WITH THE PROBLEM

While Phil changes his answer during the discussion, Max says essentially the same thing throughout. To solve the integral appropriately, one can go from " $t_0$  to  $t$ , or 0 to  $t$ , because initial time is when you throw it to some time, so just go to  $t$  instead of  $t + C$ ." Max's response is consistent with the use of integration limits in solving the problem. Importantly, though, he never talks about the mathematical form of using integration limits - there is never a " $t - t_0$ " in his talk. Instead, he talks about going "from" one time "to" another time. This suggests a view of integration as a path, with a beginning point and an ending point (a highly productive metaphor for integration in many cases). As suggested above, Phil responds to Max *as if* Max had used integrated using limits (and Max sure sounds like that's what he did).

Max uses physical meaning to suggest that the initial time can be set to 0. Max repeats this argument several times, being as stubborn in his answer as Phil is stubborn in holding on to a constant. Importantly, Max argues that "You can just mark [ $t_0$ ] as zero ... for consistency," suggesting he is thinking of some arbitrarily started stopwatch that measures time moving forward from when the object is thrown. From an epistemological perspective

[2, 3, 4, 5], Max seeks *consistency* between the physics and the math while acknowledging and leveraging that the *choice* of when the time measurement starts is *arbitrary*. He could have chosen to have the time measurement to begin at some other time, but it seems most useful to start with  $t = 0$ . (At this point, it is worth noting that neither Phil nor Max connect these constants to the  $dv$  integration which they had already completed.)

Where Phil changes his answer from  $t + C$  to  $t - t_0$ , Max objects to the original and accepts the former. But, the evidence suggests that he does so for reasons that are different from Phil's.

## DIFFERENT MEANINGS FOR A SINGLE LETTER

The solution on which Phil and Max compromise,  $t - t_0$ , is acceptable to both, but for different reasons. Phil and Max have different goals for the problem, and these goals are reflected in the meaning of  $t_0$ .

We see strong signs of Max's commitment to a physical interpretation of time when he responds to Phil's statement, "you don't know  $[t_0]$  is 0," by saying "Why, why do you need some extra time there?" It seems that he has time as a running value, as if there's a stopwatch measuring from "when you throw it" to some later time. Max seems confused that one would wish to start the clock at any time other than the moment of the throw. Thus, the quantity  $t_0$  has a very specific meaning - it is 0.

Phil, on the other hand, seeks a general equation that "anyone can use," an equation which "has to cover everything." For him, there is no value for  $t_0$ . It is an unspecified constant, just like  $C$  in his original solution. More importantly, this is the kind of unspecified constant that can be either added or subtracted. Technically, the only quantity for which such a process is mathematically allowed is one whose value is zero, but Phil does not accept that physical argument (and neither he or Max suggest the mathematical argument). We conclude that Phil is using  $t_0$  in a way that is not consistent with thinking of it as even *having* a value. For Phil,  $t_0$  acts as a placeholder. It is there to remind the user that a constant was used. But, it doesn't have a value - it can't, otherwise the equation isn't one that "anyone can use." Once someone wants to use it, they can replace the placeholder  $t_0$  with something else - why Phil won't replace it with the meaningful solution proposed by Max is unclear, since this is a specific problem, not a general one.

We suggest that Phil and Max are able to compromise on  $t - t_0$  as a solution only because they use  $t_0$  differently in their solutions. They agree on a form containing the minus sign from integration using limits, but the letters in this solution represent different things to them. For Max,

$t_0$  has a specific value, 0. For Phil, it has no value and acts only as a placeholder.

## DISCUSSION

The integral of  $dt$  seems to be simple, but Max and Phil must engage in a long debate in order to arrive at a compromise that contains a fundamental disagreement about what letters in solutions mean. (We purposefully avoid the phrase "variables in equations" to avoid making a claim about all letters being variables, all solutions being equations). During the debate, not much changes - a plus sign becomes a minus sign, a letter  $C$  becomes a  $t_0$ . The nature of the debate allows us to conclude much about Max and Phil, though.

It also allows us to ask questions about the role of "letters" in "mathematical sentences." We observe that some letters can be used as variables - Max, in particular, uses a metaphor of going from one place to the next and also talks as if thinking about a stopwatch. He seems to think of  $t$  being able to change, as if it is running forward. Phil, on the other hand, seems to be thinking in a kind of function space, where  $t$  is just  $t$ , a quantity that must *have* a constant after integration. These are very different meanings of the variable  $t$ .

Letters can also be used as constants - something that has a value, but one that stays constant. One might think that  $C$  is just such a constant, but Phil's discussion suggests that he is not seeing it as one. Max sees  $t_0$  in such a fashion, and the value is zero.

Phil uses the letters  $C$  and  $t_0$  differently from Max, namely as placeholders that ensure the generality of the mathematical solution. A placeholder does not have a value. It is in the equation to ensure that later on (when seeking a solution that works in only some cases) one goes and figures out what its value is going to be. Because it can have any value (in particular, any sign), it can be either added or subtracted. One might think of this as a parameter, in more mathematical language, but it is unclear that Phil is thinking in such a formal way. He just needs that letter there, and we can only infer the properties of that letter from what he says. (We do have evidence from other sources to suggest that the placeholder interpretation of letters is not only common to students but also to expert physicists.)

In summary, minor changes to a solution of a seemingly trivial mathematical problem uncover deep meaning not only in how two students interpret a given mathematical solution, but also in how they interpret the most basic elements of the solutions, namely the symbols and letters they use in these solutions. They listen to each other respectfully, they recognize and are able to reconstruct each other's thinking, and they continue to disagree on the mathematics. Their differences include dif-

ferent views of the connection between the mathematics and this particular problem and different views on what letters in solutions mean. These differences seem driven at least in part by their individual goals - the form of a solution that they expect. Phil wants generality, Max wants specificity - and the mathematics they use reflects these choices.

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