Student Interpretation of the Signs of Definite Integrals Using Graphical Representations

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Abstract. Physics students are expected to apply the mathematics learned in their mathematics courses to physics concepts and problems. Few PER studies have distinguished between difficulties students have with physics concepts and those with either mathematics concepts and their application or the representations used to connect the math and the physics. We are conducting empirical studies of student responses to mathematics questions dealing with graphical representations of (single-variable) integration. Reasoning in written responses could roughly be put into three major categories related to particular features of the graphs: area under the curve, position of the function, and shape of the curve. In subsequent individual interviews, we varied representational features to explore the depth and breadth of the contextual nature of student reasoning, with an emphasis on negative integrals. Results suggest an incomplete understanding of the criteria that determine the sign of a definite integral.

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INTRODUCTION

We have been exploring student difficulties with specific topics in mathematics that may affect understanding of concepts in physics. One such topic that plays a significant role in physics education is the definite integral. Our question for research here is: To what extent do students understand and reason about definite integrals using graphical representations? Our interest is with integrals and representations that have physical relevance.

Research in physics education reveals that students at the introductory level have conceptual difficulties understanding physics that involves integration [1,2]. Similarly, research in undergraduate mathematics education indicates specific issues relating to student understanding of definite integrals. Students often fail to see the relationship between a definite integral and the area under the curve [3,4], to recognize definite integrals as limits of Riemann sums [4,5], and to identify and interpret the signs of integrals of curves below the x-axis [6,7]. However, there is also evidence that reasoning about a definite integral as the area under a curve may limit students' ability to apply the integral concept [8,9].

Many facets of knowledge are needed to fully understand definite integrals; a comprehensive report on student understanding of this topic is beyond the scope of this work. Thus, we focus this report on one aspect of this topic: students' interpretations of the signs of definite integrals using graphical representations.

METHODOLOGY

One theme of our research is to identify the extent to which student difficulties with mathematical concepts affect understanding of physics concepts, particularly in the context of upper-level thermal physics. We have reported results from the administration of qualitative questions concerning the determination and comparison of the magnitudes and signs of integrals, devoid of any physics context, that were analogous to questions about thermodynamic energy transfers and changes using P-V diagrams [2]. Since these questions use notations and conventions that are more consistent with the representations used in physics than in math, we called these "physicsless physics questions" [2,10]. Based on suggestions from mathematics colleagues, as well as our own evolving research interests in student understanding of definite integrals outside of physics contexts, we modified these questions further to incorporate more thoroughly the notations and representation that are widely used in the mathematics community for representing an integral of a single-variable function. We then developed a written survey using these "analogous math" questions (Fig. 1).

The target population for our earlier research was students in undergraduate physics courses. However, as we explored the mathematical aspects in more detail, we have included students in mathematics courses as part of our study as well: we were interested to see if there was any difference in the reasoning

provided by math and physics students. The written survey was administered in calculus-based introductory physics and multivariable calculus (calc III) classes at the end of the semester after all relevant instruction. There were 97 participants in each class, with about 25 students enrolled in both courses simultaneously.

In order to probe the robustness of students' ideas about definite integrals, follow-up interviews were conducted with seven voluntary participants from the calculus-based introductory physics course. Initially, the interview participants were asked to complete a brief written questionnaire that was similar to the written survey (e.g., Fig. 1). Then, various follow-up questions were asked based on their initial lines of reasoning. The interviews were individual, semi-structured and in think-aloud format.
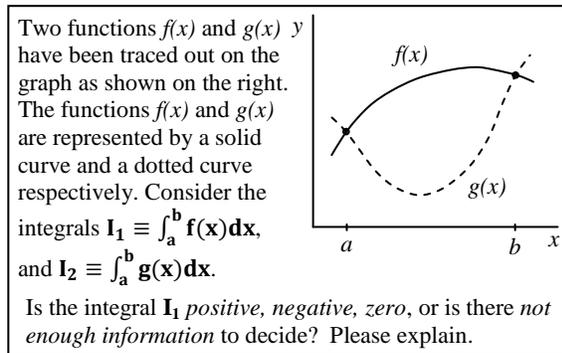

Two functions $f(x)$ and $g(x)$ have been traced out on the graph as shown on the right. The functions $f(x)$ and $g(x)$ are represented by a solid curve and a dotted curve respectively. Consider the integrals $I_1 \equiv \int_a^b f(x)dx$, and $I_2 \equiv \int_a^b g(x)dx$.

Is the integral I_1 *positive, negative, zero*, or is there *not enough information* to decide? Please explain.

FIGURE 1. The basic question asked in the written survey, and the starting question for the interviews.

RESULTS AND DISCUSSION

In both the physics and calculus classes, more than 80% of students correctly identified the sign of the integral (positive) for the question in Figure 1. However, the reasoning provided by the majority of students was either incomplete or incorrect.

Students in both physics and calculus provided a wide range of reasoning to justify their choice of the correct sign. The majority of the student reasoning falls under one of the three categories: (i) *area* under the curve, (ii) *position* of the function, or (iii) *shape* of the curve. Table 1 shows the basic categories with some examples of student reasoning. Table 2 shows the prevalence of the different lines of reasoning in the two classes (physics and calculus). A chi-squared test yields inconclusive significance for any difference in distribution of reasoning between the two classes ($\chi^2_{calculated} = 6.23$, $\chi^2_{critical}$ (at $\alpha = 0.05) = 5.99$).

The *area* reasoning in the written survey did not provide good prospects on student reasoning about the signs of definite integrals. So we conducted individual interviews to explore student use of *area* reasoning more deeply. The follow-up interview questions were focused primarily on "negative integrals."

Figure 2 shows two of the graphs used in the interviews. Figure 2a was used to ask about the sign of integral of the function below the x-axis in the direction of increasing x. Given the negative function and positive differential 'dx,' this would yield a negative integral. Figure 2b was used to ask about the sign of integral in the reverse direction, i.e. from right to left. Given the positive function, this would yield a negative integral.

TABLE 1. Student reasoning categories for determining the sign of the definite integral asked about in Fig. 1.

Category	Student reasoning
area	area under the curve is positive, area above x-axis is positive, area accumulated under the curve, etc.
position	function is in 1st quadrant, function is positive or negative, etc.
shape	the graph is concave down, $F(b) - F(a) > 0$ (Fund. Thm. Calc.), slope is increasing, etc.

TABLE 2. Prevalence of student reasoning for those students saying that the integral (see Fig. 1) was positive.

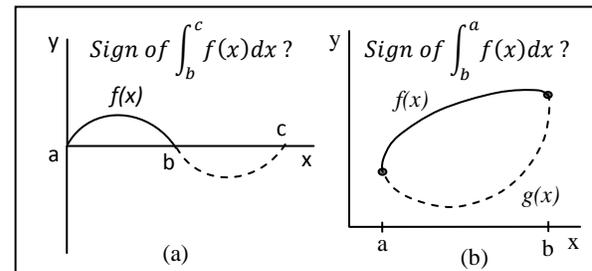| Class	Area	Position	Shape
Physics (N=97)	36	14	10
$N_{positive} = 78$	(60%)	(23%)	(16%)
Calculus (N=97)	33	26	4
$N_{positive} = 80$	(52%)	(41%)	(6%)

FIGURE 2. Follow-up interview questions for the signs of integrals of (a) a negative function towards increasing x and (b) a positive function towards decreasing x.

Student reasoning about negative integrals using area

Based on the results of the written questions and the follow-up interviews, two kinds of reasoning emerged when students who used area reasoning were asked about the signs of integrals that were negative. We provided two opportunities for negative integrals: when the function itself went below the x-axis and the

direction of integration was that of increasing *x,* and when the function was positive but the direction was that of decreasing *x,* i.e., from right to left on the graph.

Reasoning about integrals of negative functions

In the interviews, some students stated that integrals of negative functions are positive because the area is positive (or that negative area doesn't make sense). This is consistent with mathematics education literature on the topic [6,7]. However, some students used physics to justify their area-based reasoning about the sign of an integral for which the function itself was negative.

One student, Freddie, was presented with a sine-like curve (Fig. 2a) and asked about the sign of integral of a portion below the *x*-axis in the direction of increasing *x*. At first Freddie was confused because he wanted to say the integral was positive, reasoning about area as a positive spatial quantity: "In order to get negative area it is not... conceptually, looking at like a plot of land, it would be an impossibility." But then he used a physical scenario to reason through the issue: "However, we are looking at something like a voltage; voltages can very easily go negative because we only have them in reference to what we called to be ground." Freddie used physics to reason past the geometric meaning of *area* to help decide on the sign of the integral.

Another student, Abby, used the concept of work from thermodynamics to justify the sign of a positive integral based on area, even though the graphs were not labeled with physical quantities: "...finding the area underneath this graph is useful because it gives the work done in that process and I can know by if the volume gets bigger like in this process it's going to be like positive work…"

Other participants also tried to attribute physical meaning to the variables in the graph when asked about the significance of a negative integral.

Reasoning about integrals of positive functions towards decreasing x

One other way to obtain a negative integral is to have a positive function but integrate from a greater value of *x* to a lesser one. As part of the interviews, some students were asked to determine the sign of one such integral (Fig 2b). In physics this happens in multiple contexts, especially when considering thermodynamic work in a compression (volume decreases), or potential difference between two positions in an electric field.

As one might assume, students had much less facility with this type of negative integral than with the

integral of a negative function. Students provided three main arguments to reason about the sign of this kind of integral.

The first line of reasoning, which yielded a correct response, was to invoke the Fundamental Theorem of Calculus (FTC):

$$\int_a^b f(x)dx = F(b) - F(a) \qquad (1)$$

where $F(x)$ is the antiderivative of $f(x)$. Students recognized that reversing the limits reversed the terms on the right side, yielding the negative of the above quantity.

A second line of reasoning was more sophisticated, but incomplete. As an example, Simon was reasoning about the area using the Riemann sum model for a positive function integrated left-to-right: he had drawn a series of rectangles under the curve and added up these areas to get the integral (or a reasonable approximation thereof). When asked about the reverse integral, Simon said: "I feel that it should be positive because technically it shouldn't matter how you count this together right? … If you counted this way [moving his hand from right to left] or you count this way [moving his hand from left to right across the diagram] and you keep the *dx* the same, you should find the same area." Simon was adding the areas of the rectangles he drew, and those areas didn't change depending on the direction he added them. (Simon invoked the commutative property of addition – although he called it "commutalative" – to describe the property of adding in either direction yielding the same result.) Simon seemed to interpret *dx* as a distance rather than a displacement, so that it has no sign, whereas physicists would say that the *dx* in a right-to-left integral is negative, yielding the area of the rectangles in the Riemann sum to be negative.

It is worth pointing out that Simon went on to use the FTC to decide that this integral was actually negative, which supports that line of reasoning as productive in some cases. However, Simon interpreted the FTC incorrectly: "But if you're gonna go from right to left … integrating from *b* to *a* [writes $\int_b^a f(x)dx$ on the board] … if this value [the limit *b*] is larger than this value [the limit *a*] that should be a negative value [pointing towards $\int_b^a f(x)dx$ on the board] because you're gonna get a larger value here [pointing to *f(b)*] and a smaller value here [pointing to *f(a)*]." Simon confused *f(x),* the function in the integrand, with *F(x),* the antiderivative of *f(x).* Simon was not the only student to confuse these quantities. We also suspect that some *shape* reasoning may actually be this kind of reasoning; more research is needed to be sure.

Finally, as with reasoning about positive integrals, Abby used physical reasoning to deduce the sign of a reversed integral. The excerpt from Abby given above was incomplete; after saying how she could tell that the sign of thermodynamic work is positive if the volume is increasing, she states, "and then this way [right-to-left] it's going to be negative work because it's compressing and so, like that's how I know which direction to go in is by like an intuitive knowledge of what I am doing with this integral." While the invocation of a physical scenario is useful here, it's not entirely clear that Abby recognizes the mathematical reason for this, namely that *dx* (or *dV,* in this case) is actually a negative quantity. The interviewer did not pursue this line of reasoning further with students.

CONCLUSIONS

Our primary research goal was to explore students' understanding of and reasoning about definite integrals that have physical relevance, in particular about the signs of definite integrals. Written survey results indicate that most students can correctly identify the signs of definite integrals, but often with incomplete or incorrect reasoning.. We found that students' base their reasoning for the correct sign primarily on *area under the curve*, *position of the function,* or *shape of the curve*. Interviews generally support these categories, suggesting that students use them deliberately across many graph-based questions.

Interviews also allowed investigation of student understanding of two specific scenarios that yield negative definite integrals: integrals of a *negative function* in the *direction of increasing x* and integrals of a *positive function* in the *direction of decreasing x*.

Our study reveals several findings regarding the signs of negative definite integrals, in particular with students who use *area under the curve* reasoning to decide the integral signs. Many of these students state that the integral of a negative function in increasing *x* is still positive because the idea of negative area doesn't make sense to them. This finding, related to an over-reliance on the geometric interpretation of the integral as an area, is consistent with previous research in undergraduate mathematics education [6,7]. However, we also find evidence that some students who show an inability to interpret a negative integral from purely mathematical perspective often try to use a physical scenario to make sense of the meaning of the negative area.

In the case of positive functions integrated towards decreasing *x*, we find three types of arguments that students use to reason about the sign of these integrals. Some of these arguments yield correct responses, while others yield incorrect ones. Students: (i) invoke the Fundamental Theorem of Calculus, reversing the terms on the right side, which results in a (correct) negative relationship between "forward" and "backward" integrals; (ii) consider the differential *'dx'* as a fixed positive quantity, leading to a positive sign for the integral regardless of the direction of integration; and (iii) attribute physical meaning to the direction of integration, which as in the case of the integral of a negative function, seems to be a useful strategy.

We are continuing to explore this area and probe more deeply student reasoning, e.g., the extent to which students connect the physical scenario to the mathematical formalism. These results should improve our understanding of how students make sense of definite integrals mathematically and could yield insights as to instructional strategies to improve the learning and teaching of physics and the associated mathematics.

ACKNOWLEDGMENTS

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