CP Exercises

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Exercise 1:

The equation of motion for a simple, undamped pendulum is

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$$

Taking $\theta_1(t)$ and $\theta_2(t)$ as solutions to that equation, what can we say about the possible solution $\phi(t) = \theta_1(t) + \theta_2(t)$?

$$\begin{aligned}
\frac{d^2\phi}{dt^2} &= \frac{d^2}{dt^2}(\theta_1 + \theta_2) \\
&= \frac{d^2\theta_1}{dt^2} + \frac{d^2\theta_2}{dt^2} \\
&= -\frac{g}{l}\sin\theta_1 - \frac{g}{l}\sin\theta_2 \\
&\neq -\frac{g}{l}\sin(\phi)
\end{aligned}$$

Thus ϕ is not a solution and therefore the simple pendulum is not a linear system. However, in the small angle limit, where θ_1, θ_2 and ϕ are all much less than 1, the approximation $\sin\theta = \theta$ is accurate for all three angles and then gives

$$\begin{aligned}
\frac{d^2\phi}{dt^2} &= -\frac{g}{l}\sin\theta_1 - \frac{g}{l}\sin\theta_2 \\
&= -\frac{g}{l}(\theta_1 + \theta_2) \\
&= -\frac{g}{l}\phi \qquad (1)
\end{aligned}$$

And thus in this case, $\phi = \theta_1 + \theta_2$ is a solution.

Exercise 2:

(a) Any difference in the tension on the two sides of the string wrapped around the pulley will leads to a torque on the disk. The string tension on each side is proportional to the elongation of the spring on that side. The setup ensures both springs are always stretched and pulling on their end of the string and that the two tensions are equal (and let's say given by T_0) when $d = 0$ and $\theta = 0$. In this configuration, each string's length is L_0 and a change ΔL in that length makes the tension change from T_0 to $T_0 + k\Delta L$ where k is the spring constant.

As the disk/pulley assembly rotates through an angle θ, the string moves through a distance $r\theta$, where r is the pulley radius. For a given θ, the end attached to the "drive" spring (the spring attached to the drive shaft) moves up by $r\theta$ (down if θ is negative) and the end attached to the "fixed" spring (the spring fixed to the cross rod) moves down $r\theta$ (up if θ is negative).

The displacement of the bottom end of the drive spring is defined as d and is relative to its position when the length of the drive shaft is zero ($A = 0$). As the motor rotates with a non-zero A, this end of the drive spring moves up or down by the amount d (positive d is upward movement and shortens the drive spring).

Thus, the drive spring's length change is $r\theta - d$ and the string tension on the drive side becomes $T_d = T_0 + k(r\theta - d)$. The fixed spring's length change is $-r\theta$ and the string tension on the fixed side becomes $T_f = T_0 - kr\theta$. On both sides, the string pulls at right angles to the rotation axis with a moment arm given by the pulley radius, r. Thus the torque due to the string (with positive torques tending to cause accelerations in the positive direction for θ) would be given by $\tau_s = r(T_f - T_d)$; rT_f is the torque from the string pulling on the fixed spring side and $-rT_d$ is the torque from the string pulling on the drive spring side.

Because the disk is symmetric, its center of mass is on the axis of rotation and it causes no torque. The pendulum mass m is off-axis by an amount l and is directly over the axis when $\theta = 0$. It has a downward gravitational force mg acting on it and for non-zero θ, this force has a lever arm (perpendicular distance to the axis) given by $l \sin\theta$. The torque τ_g due to gravity is then $mgl \sin\theta$ and is correctly signed; positive when $\sin\theta$ is positive and negative when $\sin\theta$ is negative.

The net torque from these two sources is thus

$$\begin{aligned}
\tau_c &= \tau_s + \tau_g \\
&= r(T_f - T_d) + mgl \sin\theta \\
&= r\left[(T_0 - kr\theta) - (T_0 + kr\theta - kd)\right] + mgl \sin\theta \\
&= -2kr^2\theta + krd + mgl \sin\theta \qquad (2)
\end{aligned}$$

(b) Just as the $F_x = -dV/dx$ is the force associated with a conservative potential V, $\tau_c = -dV/d\theta$ is the torque due to a conservative potential V. Thus we should find

$$V = -\int \tau_c(d = 0)\, d\theta$$

$$
\begin{aligned}
&= \int \left(2kr^2\theta - mgl\sin\theta\right)\, d\theta \\
&= kr^2\theta^2 + mgl\cos\theta
\end{aligned}
$$

The integration constant can be taken as zero as any overall constant in the potential will have no effect on the motion.

(c) Minima and maxima in the potential are where its derivative is zero, i,.e., where the net torque is zero.

$$
\begin{aligned}
0 &= \left.\frac{dV}{d\theta}\right|_{\theta=\theta_e} \\
&= \left.\frac{d}{d\theta}\left(kr^2\theta^2 + mgl\cos\theta\right)\right|_{\theta=\theta_e} \\
&= 2kr^2\theta_e - mgl\sin\theta_e
\end{aligned}
$$

This has a simple solution $\theta_e = 0$, but it may have additional solutions with $\theta_e \neq 0$ where

$$
\frac{\sin\theta_e}{\theta_e} = \frac{2kr^2}{mgl}
$$

Note that the left side has a maximum value of unity when $\theta_e = 0$. So the right side must be less than unity to have any additional solutions beyond $\theta_e = 0$.

Exercise 3:

(a) The main equation is that the net torque is the moment of inertia times the angular acceleration. The net torque is that due to conservative forces τ_c given by Eq. 2 and the torque due to magnetic damping and axle friction,

$$
\tau_f = -b\,\omega - b'\mathrm{sgn}(\omega)
$$

That is,

$$
\begin{aligned}
I\frac{d^2\theta}{dt^2} &= \tau_c + \tau_f \\
&= -2kr^2\theta + krd + mgl\sin\theta - b\,\omega - b'\mathrm{sgn}(\omega) \qquad (3)
\end{aligned}
$$

The definition of the angular velocity ω is

$$
\frac{d\theta}{dt} = \omega \qquad (4)
$$

Note that the drive displacement d is given by

$$d = A\cos\phi \tag{5}$$

where ϕ is the angle of the motor shaft (assuming straight up is $\phi = 0$). Then if the motor rotates at a constant angular velocity Ω, the drive shaft angle is given by $\phi = \Omega t$ (assuming $\phi = 0$ at $t = 0$) and gives

$$\frac{d\phi}{dt} = \Omega$$

Using Eqs. 4 and 5 in Eq. 3 and dividing through by the moment of inertia gives

$$\frac{d\omega}{dt} = -\frac{2kr^2}{I}\theta + \frac{krA}{I}\cos\phi + \frac{mgl}{I}\sin\theta - \frac{b}{I}\omega - \frac{b'}{I}\mathrm{sgn}(\omega) \tag{6}$$

or in the dot notation:

$$\begin{aligned}
\dot{\theta} &= \omega \\
\dot{\omega} &= -\kappa\theta + \mu\sin\theta + \epsilon\cos\phi - \Gamma\omega - \Gamma'\mathrm{sgn}(\omega) \\
\dot{\phi} &= \Omega
\end{aligned}$$

where

$$\begin{aligned}
\kappa &= 2kr^2/I \\
\mu &= mgl/I \\
\epsilon &= krA/I \\
\Gamma &= b/I \\
\Gamma' &= b'/I
\end{aligned}$$

Exercise 4:

(a) The equation for the potential V is

$$V = kr^2\theta^2 + mgl\,\cos\theta$$

Expanding any function $f(x)$ about some point x_0 via a Taylor expansion is $f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + \ldots$ Here we expand $V(\theta)$ about $\theta_0 = \pm\theta_e$ keeping only up to the quadratic term. We can drop the linear term because the $V'(\theta_0) = 0$ for $\theta_0 = \pm\theta_e$ because the definition of θ_e is precisely where that derivative vanishes. Thus

$$\begin{aligned} V(\theta) &= V(\theta_0) + \frac{1}{2}\left.\frac{d^2V(\theta)}{d\theta^2}\right|_{\theta=\theta_0}(\theta-\theta_0)^2 \\ &= kr^2\theta_0^2 + mgl\,\cos\theta_0 + \frac{1}{2}(2kr^2 - mgl\cos\theta_0)(\theta-\theta_0)^2 \\ &= kr^2\theta_e^2 + mgl\,\cos\theta_e + \frac{1}{2}(2kr^2 - mgl\cos\theta_e)(\theta-\theta_0)^2 \end{aligned}$$

where $\theta_0 = \pm\theta_e$ is substituted in those places where the result is independent of the sign of θ_0.

(b) The torque due to the conservative forces (from the spring and gravity) is just

$$\begin{aligned} \tau_c &= -\frac{dV}{d\theta} \\ &= -(2kr^2 - mgl\cos\theta_e)(\theta-\theta_0) \end{aligned}$$

The net torque including the magnetic damping $\tau_f = -b\,\omega$ (but neglecting axle friction) and including the torque due to the drive $\tau_d = krd = kA\cos\phi = krA\cos\Omega t$

$$\tau = -(2kr^2 - mgl\cos\theta_e)(\theta-\theta_0) + krA\cos\Omega t - b\,\omega$$

Noting that $\theta' = \theta - \theta_0$ and thus $\omega = d\theta/dt = d\theta'/dt$ and setting the angular acceleration $\ddot{\theta}'$ equal to the net torque divided by the moment of inertia gives:

$$\begin{aligned} \ddot{\theta}' &= -\frac{1}{I}\left[(2kr^2 - mgl\cos\theta_e)\theta' + krA\cos\Omega t - b\dot{\theta}'\right] \\ &= -(\kappa - \mu\cos\theta_e)\theta' + \epsilon\cos\Omega t - \Gamma\dot{\theta}' \end{aligned}$$

Rearranging and setting $\Omega_0^2 = \kappa - \mu\cos\theta_e$ gives

$$\ddot{\theta}' + \Gamma\dot{\theta}' + \Omega_0^2\theta' = \epsilon\cos\Omega t$$

Exercise 5:

The equation of motion is given as

$$\ddot{\theta} + \Gamma\dot{\theta} + \Omega_0^2\theta = 0 \tag{7}$$

and we are to check the validity of the solution

$$\theta = Ce^{-\Gamma t/2}\cos(\Omega' t + \delta) \tag{8}$$

One could simply show this solution satisfies the differential equation. Here, we will assume a solution of the form

$$\theta = Ce^{-\gamma t}\cos(\Omega' t + \delta)$$

and see what conditions arise on the constants C, γ, Ω' and δ appearing in it. Its first derivative is

$$\dot{\theta} = Ce^{-\gamma t}\left(-\gamma\cos(\Omega' t + \delta) - \Omega'\sin(\Omega' t + \delta)\right)$$

Its second derivative is

$$\begin{aligned}\ddot{\theta} &= Ce^{-\gamma t}\left(\gamma^2\cos(\Omega' t + \delta) + \gamma\Omega'\sin(\Omega' + \delta) + \gamma\Omega'\sin(\Omega' t + \delta) - \Omega'^2\cos(\Omega' t + \delta)\right)\\ &= Ce^{-\gamma t}\left(2\gamma\Omega'\sin(\Omega' t + \delta) + \left(\gamma^2 - \Omega'^2\right)\cos(\Omega' t + \delta)\right)\end{aligned}$$

Putting these last three equations into the equation of motion gives

$$\begin{aligned}&Ce^{-\gamma t}\left(2\gamma\Omega'\sin(\Omega' t + \delta) + \left(\gamma^2 - \Omega'^2\right)\cos(\Omega' t + \delta)\right)\\ &\quad + \Gamma\left(Ce^{-\gamma t}\left(-\gamma\cos(\Omega' t + \delta) - \Omega'\sin(\Omega' t + \delta)\right)\right)\\ &\quad + \Omega_0^2\, Ce^{-\gamma t}\cos(\Omega' t + \delta) = 0\end{aligned}$$

Canceling the $Ce^{\gamma t}$ and collecting terms gives

$$\Omega'\left(2\gamma - \Gamma\right)\sin(\Omega' t + \delta) + \left(\gamma^2 - \Omega'^2 - \gamma\Gamma + \Omega_0^2\right)\cos(\Omega' t + \delta) = 0$$

As this must hold for all values of t, both coefficients multiplying the oscillating factors $\sin(\Omega' t + \delta)$ and $\cos(\Omega' t + \delta)$ must be zero.

$$\begin{aligned}0 &= \Omega'\left(2\gamma - \Gamma\right)\\ 0 &= \left(\gamma^2 - \Omega'^2 - \gamma\Gamma + \Omega_0^2\right)\end{aligned}$$

The first of these two equations gives

$$\gamma = \frac{\Gamma}{2}$$

And using this in the second equation gives

$$0 \;=\; \left(\frac{\Gamma}{2}\right)^2 - \Omega'^2 - \frac{\Gamma^2}{2} + \Omega_0^2$$

or

$$\Omega'^2 \;=\; \Omega_0^2 - \frac{\Gamma^2}{4} \tag{9}$$

Thus giving Eq. 8 as a solution (with Eq. 9 for the free oscillation frequency Ω'). The differential equation does not give any conditions on the amplitude factor C or on the phase constant δ which are determined by initial conditions on the angle $\theta(0)$ and the angular velocity $\omega(0)$ at $t = 0$

(b) The general solution to the driven harmonic oscillator equation

$$\ddot{\theta} + \Gamma\dot{\theta} + \Omega_0^2\theta \;=\; \epsilon\cos\Omega t \tag{10}$$

can be expressed

$$\theta \;=\; \theta_p + \theta_h \tag{11}$$

where θ_h is the (homogeneous) solution to the undriven system and given by Eq. 8 with Eq. 9 and θ_p is any particular solution to Eq. 10. To see this, try Eq. 11 in Eq. 10

$$\begin{aligned}
\frac{d^2}{dt^2}(\theta_p + \theta_h) + \Gamma\frac{d}{dt}(\theta_p + \theta_h) + \Omega_0^2(\theta_p + \theta_h) &= \epsilon\cos\Omega t \\
\ddot{\theta}_p + \Gamma\dot{\theta}_p + \Omega_0^2\theta_p + \ddot{\theta}_h + \Gamma\dot{\theta}_h + \Omega_0^2\theta_h &= \epsilon\cos\Omega t \\
\ddot{\theta}_p + \Gamma\dot{\theta}_p + \Omega_0^2\theta_p &= \epsilon\cos\Omega t
\end{aligned}$$

where in getting to the last line the fact that θ_h satisfies Eq. 7 was used. We will try as the particular solution:

$$\theta_p = C'\cos(\Omega t + \delta') \tag{12}$$

and see what conditions arise on C' and δ'. Notice the form of Eq. 12. It is constant amplitude oscillations at the drive frequency Ω and is offset in phase from the drive waveform $\epsilon\cos\Omega t$ by the amount δ'. The derivative of Eq. 12 is $\dot{\theta}_p = -\Omega C'\sin(\Omega t + \delta')$ and the second derivative is $\ddot{\theta}_p = -\Omega^2 C'\cos(\Omega t + \delta')$. Using these in Eq. 10 gives

$$\begin{aligned}
\ddot{\theta}_p + \Gamma\dot{\theta}_p + \Omega_0^2\theta_p &= \epsilon\cos\Omega t \\
-\Omega^2 C'\cos(\Omega t+\delta') - \Gamma\Omega C'\sin(\Omega t+\delta') + \Omega_0^2 C'\cos(\Omega t+\delta') &= \epsilon\cos\Omega t \\
(\Omega_0^2-\Omega^2)\cos(\Omega t+\delta') - \Gamma\Omega\sin(\Omega t+\delta') &= \frac{\epsilon}{C'}\cos\Omega t \\
(\Omega_0^2-\Omega^2)\left(\cos\Omega t\cos\delta' - \sin\Omega t\sin\delta'\right) - \Gamma\Omega\left(\sin\Omega t\cos\delta' + \cos\Omega t\sin\delta'\right) &= \frac{\epsilon}{C'}\cos\Omega t \\
\left((\Omega_0^2-\Omega^2)\cos\delta' - \Gamma\Omega\sin\delta' - \frac{\epsilon}{C'}\right)\cos\Omega t - \left((\Omega_0^2-\Omega^2)\sin\delta' + \Gamma\Omega\cos\delta'\right)\sin\Omega t &= 0 \qquad (13)
\end{aligned}$$

This equation must be satisfied for all t and thus the coefficients of the $\sin\Omega t$ and $\cos\Omega t$ oscillatory terms must both be zero. For the $\sin\Omega t$ term this gives:

$$(\Omega_0^2-\Omega^2)\sin\delta' + \Gamma\Omega\cos\delta' \;=\; 0$$

or

$$\tan\delta' \;=\; \frac{-\Gamma\Omega}{\Omega_0^2-\Omega^2}$$

With $\sin^2\delta' + \cos^2\delta' = 1$, this gives

$$\begin{aligned}
\sin\delta' &= \frac{-\Gamma\Omega}{\sqrt{(\Omega_0^2-\Omega^2)^2+\Gamma^2\Omega^2}} \\
\cos\delta' &= \frac{\Omega_0^2-\Omega^2}{\sqrt{\left(\Omega_0^2-\Omega^2\right)^2+\Gamma^2\Omega^2}}
\end{aligned}$$

Using these last two expresions in the $\cos\Omega t$ term of Eq. 13 gives

$$\begin{aligned}
(\Omega_0^2-\Omega^2)\frac{\Omega_0^2-\Omega^2}{\sqrt{\left(\Omega_0^2-\Omega^2\right)^2+\Gamma^2\Omega^2}} - \Gamma\Omega\frac{-\Gamma\Omega}{\sqrt{\left(\Omega_0^2-\Omega^2\right)^2+\Gamma^2\Omega^2}} &= \frac{\epsilon}{C'} \\
\frac{\left(\Omega_0^2-\Omega^2\right)^2+\Gamma^2\Omega^2}{\sqrt{\left(\Omega_0^2-\Omega^2\right)^2+\Gamma^2\Omega^2}} &= \frac{\epsilon}{C'} \\
\sqrt{\left(\Omega_0^2-\Omega^2\right)^2+\Gamma^2\Omega^2} &= \frac{\epsilon}{C'}
\end{aligned}$$

or

$$C = \frac{\epsilon}{\sqrt{\left(\Omega_0^2-\Omega^2\right)^2+\Gamma^2\Omega^2}}$$

Alternate solution using phasors

Both parts (a) and (b) are much more easily solved using phasors. Euler's equation is

$$e^{i\phi} = \cos\phi + i\sin\phi$$

With $z = x + i\,y$ representing a general complex quantity, $\Re\{z\} = x$ is the function that takes its real part and $\Im\{z\} = y$ takes its imaginary part.

(a) These relations are used, for example, to express a trial solution for part (a) in the form

$$\theta \;=\; \Re\left\{\boldsymbol{C}e^{\boldsymbol{\omega}t}\right\} \tag{14}$$

where both $\boldsymbol{C}$ and $\boldsymbol{\omega}$ are now complex quantities whose values are to be determined. Substituting the trial solution into Eq. 7 gives:

$$\begin{aligned}\left[\frac{d^2}{dt^2} + \Gamma\frac{d}{dt} + \Omega_0^2\right]\Re\left\{\boldsymbol{C}e^{\boldsymbol{\omega}t}\right\} &= 0\\ \Re\left\{\left[\frac{d^2}{dt^2} + \Gamma\frac{d}{dt} + \Omega_0^2\right]\boldsymbol{C}e^{\boldsymbol{\omega}t}\right\} &= 0\end{aligned} \tag{15}$$

Taking the real part can be moved to the front because the differential equation is all real. Now evaluating the derivatives is very simple and leads to

$$\Re\left\{\left[\boldsymbol{\omega}^2 + \Gamma\boldsymbol{\omega} + \Omega_0^2\right]\boldsymbol{C}e^{\boldsymbol{\omega}t}\right\} \;=\; 0$$

Because the term $\boldsymbol{C}e^{\boldsymbol{\omega}t}$ has oscillating real and imaginary components, the expression above can only be zero for all times if the factor in square brackets is identically zero for both its real and imaginary parts. This gives an algebraic equation (the quadratic formula)

$$\boldsymbol{\omega}^2 + \Gamma\boldsymbol{\omega} + \Omega_0^2 \;=\; 0$$

which has the solution

$$\begin{aligned}\boldsymbol{\omega} &= -\frac{\Gamma}{2} \pm \sqrt{\Gamma^2/4 - \Omega_0^2}\\ &= -\frac{\Gamma}{2} \pm i\sqrt{\Omega_0^2 - \Gamma^2/4}\\ &= -\frac{\Gamma}{2} \pm i\Omega_0'\end{aligned}$$

where

$$\Omega_0' = \sqrt{\Omega_0^2 - \frac{\Gamma^2}{4}}$$

Which when plugged back into the solution (with the constant $\boldsymbol{C}$ expressed in the form $Ae^{i\delta}$

$$\begin{aligned}
\theta &= \Re\left\{\boldsymbol{C}e^{\boldsymbol{\omega}t}\right\} \\
&= \Re\left\{Ae^{i\delta}e^{-(\Gamma/2 \pm i\Omega_0')t}\right\} \\
&= \Re\left\{Ae^{-\Gamma t/2 \pm i(\Omega_0' t \mp \delta)}\right\} \\
&= Ae^{-\Gamma t/2}\Re\left\{e^{\pm i(\Omega_0' t \mp \delta)}\right\} \\
&= Ae^{-\Gamma t/2}\cos(\Omega_0' t + \delta)
\end{aligned}$$

(b) Trying the following particular solution

$$\theta_p = \Re\left\{\boldsymbol{C}e^{i\Omega t}\right\}$$

in the driven oscillator equation gives

$$\begin{aligned}
\left[\frac{d^2}{dt^2} + \Gamma\frac{d}{dt} + \Omega_0^2\right]\theta_p &= \epsilon\cos\Omega t \\
\left[\frac{d^2}{dt^2} + \Gamma\frac{d}{dt} + \Omega_0^2\right]\Re\left\{\boldsymbol{C}e^{i\Omega t}\right\} &= \epsilon\Re\left\{e^{i\Omega t}\right\} \\
\Re\left\{\left[\frac{d^2}{dt^2} + \Gamma\frac{d}{dt} + \Omega_0^2\right]\boldsymbol{C}e^{i\Omega t}\right\} &= \epsilon\Re\left\{e^{i\Omega t}\right\} \\
\Re\left\{\left[\frac{d^2}{dt^2} + \Gamma\frac{d}{dt} + \Omega_0^2\right]\boldsymbol{C}e^{i\Omega t} \right. &= \left. \epsilon e^{i\Omega t}\right\}
\end{aligned}$$

In the third line, the real part can be pulled out front because the differentiation operations are all real and in the fourth line the real part is pulled around the whole equation because if that equation is satisfied, both the real and imaginary parts would be satisfied. Solving that equation will actually be straightforward.

$$\begin{aligned}
\left[\frac{d^2}{dt^2} + \Gamma\frac{d}{dt} + \Omega_0^2\right]\boldsymbol{C}e^{i\Omega t} &= \epsilon e^{i\Omega t} \\
\left[-\Omega^2 + i\Gamma\Omega + \Omega_0^2\right]\boldsymbol{C}e^{i\Omega t} &= \epsilon e^{i\Omega t} \\
\left[\Omega^2 - \Omega^2 + i\Gamma\Omega\right]\boldsymbol{C} &= \epsilon
\end{aligned}$$

Where in the last line, the $e^{i\Omega t}$ was canceled from both sides. Now solving for $\boldsymbol{C}$ proceeds as follows

$$\begin{aligned}\boldsymbol{C} &= \frac{\epsilon}{\Omega_0^2 - \Omega^2 + i\Gamma\Omega} \\ &= \frac{\epsilon}{\Omega_0^2 - \Omega^2 + i\Gamma\Omega} \cdot \frac{\Omega_0^2 - \Omega^2 - i\Gamma\Omega}{\Omega_0^2 - \Omega^2 - i\Gamma\Omega} \\ &= \epsilon \frac{\Omega_0^2 - \Omega^2 - i\Gamma\Omega}{\left(\Omega_0^2 - \Omega^2\right)^2 + \Gamma^2\Omega^2}\end{aligned} \tag{16}$$

Now the denominator is all real. To express $\boldsymbol{C} = C' e^{i\delta'}$, one uses $\tan\delta' = \Im\boldsymbol{C}/\Re\boldsymbol{C}$. Since it is a ratio, the denominator drops out leaving

$$\tan\delta' = \frac{-\Gamma\Omega}{\Omega_0^2 - \Omega^2}$$

The magnitude C' is most easily determined from Eq. 16 above. It is simply ϵ (a real) divided by the magnitude of the denominator. The magnitude of a complex $z = x + i\,y$ is simply $\sqrt{x^2 + y^2}$ giving

$$C' = \frac{\epsilon}{\sqrt{\left(\Omega_0^2 - \Omega^2\right)^2 + \Gamma^2\Omega^2}}$$

Selected Comprehension Questions

Comprehension Question 1:

For undamped undriven harmonic oscillations θ executes simple harmonic motion

$$\theta = A\cos\Omega_0 t$$

where the phase constant has been chosen to be zero. With this time dependence for θ, the angular velocity becomes

$$\begin{aligned}\omega &= \frac{d\theta}{dt} \\ &= -A\Omega_0 \sin\Omega_0 t\end{aligned}$$

to see what a phase plot would look like we can note that $\sin^2\Omega_0 t + \cos^2\Omega_0 t = 1$ for all t or

$$\frac{\theta^2}{A^2} + \frac{\omega^2}{A^2\Omega_0^2} = 1 \tag{17}$$

which is the equation of an ellipse with the horizontal (θ) radius equal to A and the vertical (ω) radius equal to $A\Omega_0$. Thus to reshape the ellipse into a circle one only needs to scale the lengths accordingly. With a given angle plotted along the x-axis according to any scale factor α (in units of say centimeters per radian), the y-axis scale factor for plotting angular velocities would have to be α/Ω_0 (in units of centimeters per rad/s). A phase point travels clockwise around the ellipse on a normally arrange coordinate system (with increasing values plotted upward and to the right). Consider a point at the top of the ellipse where $\theta = 0$ and $\omega > 0$. Since $\omega = d\theta/dt$ is positive, θ must be increasing and moving rightward, i.e., clockwise. The time to go around the ellipse is determined by the period of the oscillations $T = 2\pi/\Omega_0$, since Ω_0 is the angular frequency of the oscillations. The amplitude A of the angular oscillations depends on the initial conditions. The sketch should show two ellipses with the same aspect ratio and centers, but with different sizes.

Comprehension Question 2:

Damping causes an elliptical trajectory to decay spirally into the center. The decay can be exponential with only magnetic damping to a something a bit more linear in the case of axle friction. The sketches should show two spirals decaying to the origin, where all trajectories will ultimately terminate.

Comprehension Question 3:

(a) Keep in mind the definition requires the slope od the $\log N$ vs. $\log M$ graph be taken in the large M limit, where the box size has been made infinitesimally small. If the attractor is a finite number of points in three dimensional phase space, then as the box size decreases, at some point each such point will be in one box only. Any further decrease in the box size will leave the number of boxes occupied unchanged and the slope of the $\log N$ vs. $\log M$ graph will be zero. If the attractor is any finite number of lines in phase space then the number of boxes occupied will increase as the box size decreases. Imagine the boxes have been made small enough that for any box containing any part of any line inside it is, the line is affectively straight through the box. Cutting the box size in half, there would now be twice as many boxes occupied. A direct proportionality like this results in the $\log N$ vs. $\log M$ graph having a slope of 1. If the attractor is any finite number of areas in phase space, the number of boxes occupied will again increase as the box size decreases. Imagine the boxes have been made small enough that for any box containing any part of any area inside it is, the area is affectively flat through the box. Cutting the box size in half, there would now be four times as many boxes occupied. A quadratic proportionality like this results in the $\log N$ vs. $\log M$ graph having a slope of 2. If the attractor is any finite number of volumes in phase space, the number of boxes occupied will again increase as the box size decreases. Imagine the boxes have been made small enough that for any box containing any part of any volume, the volume completely fills the box. Cutting the box size in half, there would now be eight boxes occupied. A cubic proportionality like this results in the $\log N$ vs. $\log M$ graph having a slope of 3.

(b) When there are two or more such objects in phase space, as the boxes get smaller and smaller, sooner or later the object with the largest dimensionality will occupy the largest number of boxes (by far) and will dominate the behavior of the $\log N$ vs. $\log M$ graph.

Comprehension Question 4:

The derivation proceeds as follows.

$$\begin{aligned}
\alpha &= -\Gamma\omega - \Gamma' \operatorname{sgn}\omega - \kappa(\theta_m + \delta\theta - \theta_0) + \mu\sin(\theta_m + \delta\theta) + \epsilon\cos(\phi_m + \delta\phi) \\
&= -\Gamma\omega - \Gamma' \operatorname{sgn}\omega - \kappa(\theta_m + \delta\theta - \theta_0) + \mu(\sin\theta_m \cos\delta\theta + \cos\theta_m \sin\delta\theta) \\
&\quad\; +\epsilon(\cos\phi_m \cos\delta\phi - \sin\phi_m \sin\delta\phi) \\
&= -k(\delta\theta - \theta_0) - \Gamma\omega - \Gamma' \operatorname{sgn}\omega - \kappa\theta_m + \mu\cos\delta\theta\sin\theta_m + \mu\sin\delta\theta\cos\theta_m \\
&\quad\; +\epsilon\cos\delta\phi\cos\phi_m - \epsilon\sin\delta\phi\sin\phi_m
\end{aligned}$$