Meaning Behind Mathematical Moves

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Abstract: We analyze the discourse of faculty presenting derivations in which they manipulate mathematical equations to illuminate a physical principle. Observations are interpreted through a lens of *symbolic forms*, conceptual and contextual meanings that are embedded in the equation. When an equation is manipulated (e.g. bringing terms to one side or another), different forms are emphasized, changing the meaning of the equation. We argue that this framework can make explicit the faculty motivations for the moves, and present two manipulations that appear to have distinctly different reasons. The first manipulation brings about a change in context from a physics to a mathematical frame. In the second, a thematic manipulation --- grouping all terms of a common variable --- reveals an important conceptual point about a driven harmonic oscillator. While there is direct evidence from the observed faculty to support the inference of motivation, in neither case is the reasoning made clear to the students. The study of discourse represents a new direction in which physics education researchers can study and inform the classroom.

Physics equations contain conceptual and contextual meanings. Manipulations change these meanings, obscuring some and illuminating others.

Symbolic Forms: conceptual and contextual meaning in equations

Sherin (*Cog.& Inst.*, **19**, 2001) defines *symbolic forms* as elemental relations associating conceptual schema with a pattern of symbols.

Symbolic forms are context-dependent

Two mathematically identical equations may have very different physics schema. Consider the equations:

$$\vec{v}_f = \vec{v}_o + \vec{a}t$$

$$\vec{F}_{net} = m\vec{g} + k\vec{x}$$

Mathematically, both equate a quantity with the sum of two terms. Physicists add conceptual context: the first is a kinematics equation and the second a Newton's 2nd Law equation. This gives different meaning to the added terms. The velocity has an initial value v_o and changes due to an acceleration a, a"**base+change**".

Concepts of "base" and "change" are not relevant in the second equation. The net force F_{net} is comprised of two terms, each a force in its own right. One does not think of either force as representing a "change" in the net force, and so the base+change form does not apply. Rather, this is a "**sum of the parts**" form.

Symbolic forms (Sherin, 2011)

Cluster	*Symbolic form*	*Symbol pattern*
Competing terms cluster	Competing terms	□ ± □ ± □ …
	Opposition	□ – □
	Balancing	□ = □
	Canceling	0 = □ – □
Terms are amounts cluster	Parts-of-a-whole	[□ + □ + □ …]
	Base ± change	[□ ± Δ]
	Whole – part	[□ – □]
	Same amount	□ = □
Dependence cluster	Dependence	[… x …]
	No dependence	[…]
	Sole dependence	[… x …]
Coefficient cluster	Coefficient	[x □]
	Scaling	[n □]
Multiplication cluster	Intensive–extensive	x × y
	Extensive–extensive	x × y
Proportionality cluster	Prop+	$\left[\frac{\ldots x \ldots}{\ldots}\right]$
	Prop–	$\left[\frac{\ldots}{\ldots x \ldots}\right]$
	Ratio	$\left[\frac{x}{y}\right]$
	Canceling (b)	$\left[\frac{\ldots x \ldots}{\ldots x \ldots}\right]$
Other	Identity	x = …
	Dying away	$\left[e^{-x\ldots}\right]$

What story do these mathematical moves tell?

$$m\ddot{\vec{x}} = -k\vec{x} - c\dot{\vec{x}} + F(t)$$

$$m\ddot{\vec{x}} + k\vec{x} + c\dot{\vec{x}} = F(t)$$

$$F(t) = F_o \cos\omega t, c = 0$$

$$m\ddot{\vec{x}} + k\vec{x} = F(t)$$

$$\vec{x} = A\cos(\omega t + \phi)$$

$$-m\omega^2 A\cos(\omega t + \phi) + A\cos(\omega t + \phi) = F\cos\omega t$$

$$A(k - m\omega^2) = \frac{F_o \cos\omega t}{\cos(\omega t + \phi)}$$

$$A = \frac{F_o}{k - m\omega^2}$$

Methodological Details

- qualitative case study
- emergent, non-rubric based analysis
- analyses presented to four independent education researchers to confirm plausibility and logic
- validation sought in post-observation interviews and/or verbal discourse analysis
- **Observer biases:** both researchers were physicists familiar with content who assumed conceptual and contextual meaning within the math

Classroom Context

- junior Classical Mechanics course on driven harmonic oscillator
- traditional lecture-type classroom (students in rows facing front)
- experienced Senior Lecturer, had taught the course the previous academic year and consistently received excellent evaluations
- very little verbal discourse until the very end

Act I: Changing forms changing frames

force in opposite direction of displacement

damping force opposes motion

$$m\ddot{x} = -kx - c\dot{x} + F(t)$$

Net force produces a proportional acceleration

forces can be summed into an important net force

In the initial form, the equation emphasizes the net force as the sum of its parts, which include both a restoring and a damping force, and that the result of this force is to produce an acceleration.

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

This combination of positions, velocities, and accelerations ***always equals applied force***

Rearranging the equation obscures the physics concepts: forces are no longer added to a single net force and concepts of restoring and damping are obscured. Instead, the equation is now in a standard form that suggests a mathematical solution to an inhomogeneous differential equation: all terms involving the variable x (and derivatives) are grouped together and the time-dependent function is isolated on the right.

This interpretation was validated in a follow-up interview with the course instructor, who explicitly stated that he did this to "set up the problem to be solved as a differential equation."`

Denouement: Revealing new concepts

$$-m\omega^2 A\cos(\omega t + \phi) + A\cos(\omega t + \phi) = F\cos\omega t$$

Written this way the equation asks one to **balance** two terms that oscillate as $\omega t+\varphi$ with a term that oscillates as ωt. From this, the instructor *could* argue that $\varphi=0$ ***He chooses not to.*** Instead, he **isolates all time-dependent terms**

$$A(k - m\omega^2) = \frac{F_o \cos\omega t}{\cos(\omega t + \phi)}$$

This emphasizes the equality between a time-independent and (on the surface) time-dependent quantity, the compound concept he wishes to convey. (We note that the *(k-$m\omega^2$)* term remains grouped with A.) The motivation is also seen in analysis of the verbal discourse that immediately follows the manipulation. Boldface indicates verbal cues to separate smaller concepts.

Right. *So if we do that this (LHS) is obviously some type of constant. It is not dependent on time. Which means that in order for this (RHS) to be a solution this (RHS) can also not be a function of time.* ***Alright.*** *So that means you've got two possible solutions. Either you have phi equals 0 or phi equals pi. And if phi equals 0 this ratio equals 1 and if phi equals pi this ratio equals minus one.* ***Right.*** *So that's the only two solutions that you can get.*