# Students’ dynamic geometric reasoning about quantum spin-1/2 states

Hunter G. Close, Catherine C. Schiber, Eleanor W. Close, and David Donnelly

Department of Physics, Texas State University–San Marcos

TEXAS STATE UNIVERSITY SAN MARCOS

## DATA AND METHODS

### Course context

- 1st semester upper-division QM course
- Spin-first approach[7] with spin as example of Hilbert space with a small number of dimensions, with applications to fundamentals like uncertainty, complex phase, and commutation

### Oral Examination

- 1st of two, this one ~1/3 of the way through the course
- 13 students
- Students were provided the following framework for preparing for the exam

| *Framework for oral exam* |
|---|
| 1. Find an unknown spin-1/2 state from probabilities of spin measurements in the x, y, and z directions. |
| 2. For a set of quantum states, determine which would generate the same set of probabilities for spin measurements and which would generate different sets of probabilities, without needing to perform direct calculations of these probabilities. |

### Method of video analysis

- Student solutions to problem 1 were almost entirely algebraic and were not analyzed in this study
- Focus of observation was on student use of and interaction with NPDs, especially gesture

### Oral exam problem analyzed here

**The “Distinguishable states” problem**

Which of these quantum states would generate the same set of probabilities for spin-1/2 measurements (in any/all directions) as each other, and which would generate different sets of probabilities?

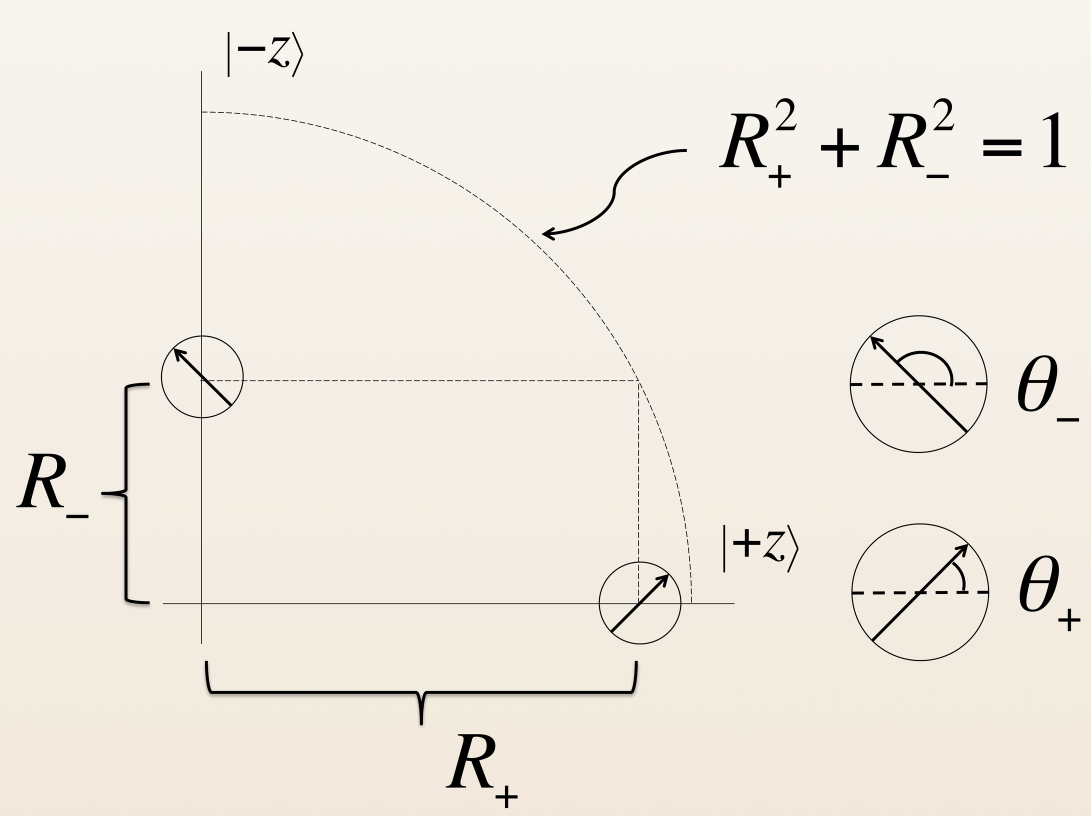
| $\lvert\psi\rangle = c_1\lvert+z\rangle + +c_2\lvert-z\rangle$ | $c_1$ | $c_2$ |
|---|---|---|
| A | $\frac{1}{\sqrt{2}}$ | $\frac{1+i}{2}$ |
| B | $\frac{1+i}{2}$ | $\frac{1}{\sqrt{2}}$ |
| C | $\frac{1+i}{2}$ | $\frac{1+i}{2}$ |
| D | $\frac{1}{\sqrt{2}}$ | $\frac{1}{\sqrt{2}}$ |
| E | $\frac{i}{\sqrt{2}}$ | $\frac{1+i}{2}$ |

These five states A-E vary only in relative phase and absolute phase, not in relative magnitude of the coefficients. Five additional states F-K were provided that had varying relative magnitude. Most students did not have time to address F-K.

## BACKGROUND: Spin-1/2 and Nested Phasor Diagrams (*NPDs*)

$$|\psi\rangle = R_+ e^{i\theta_+}|+z\rangle + R_- e^{i\theta_-}|-z\rangle$$

Any spin-1/2 state can be written algebraically like this.




Using Nested Phasor Diagrams, any spin-1/2 state can be diagrammed like this.

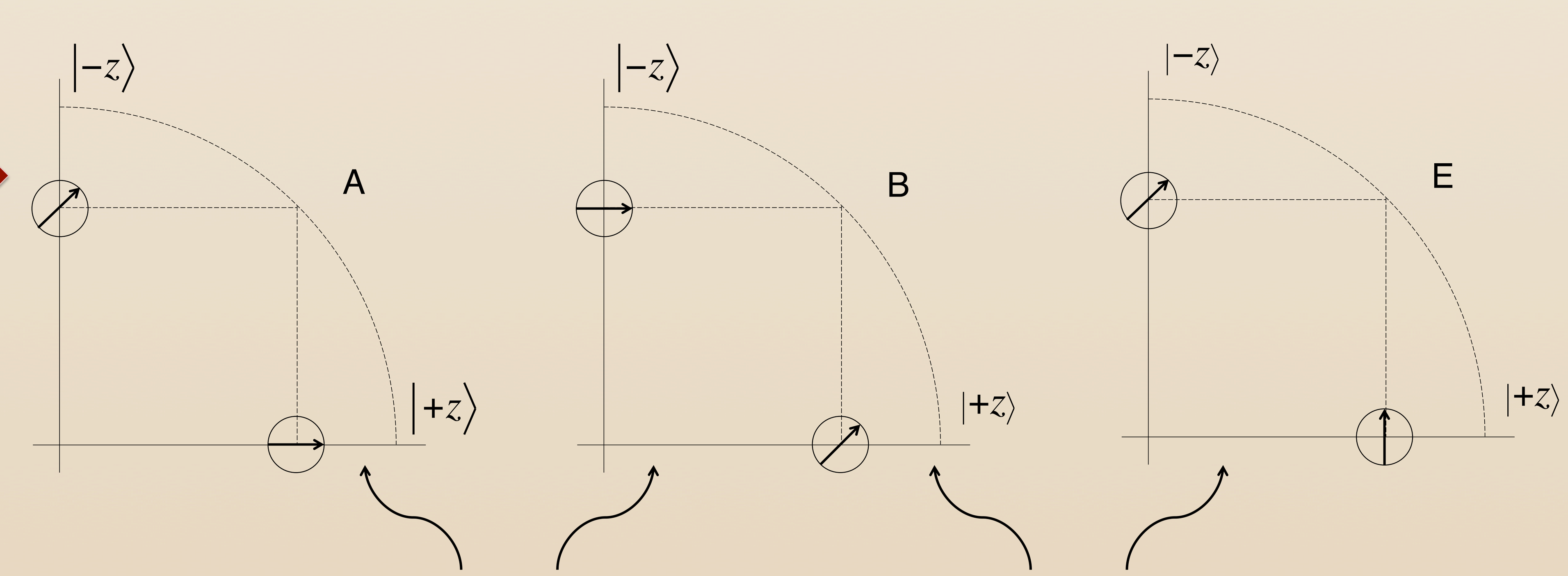
The NPD displays the magnitudes and phase angles of the complex coefficients (four real numbers), shows the normalization constraint (circular arc), and shows the orthogonality relation between different components of the state (perpendicular axes).

Probabilities of measuring spin up or down in various directions in lab space depend in general on the relative magnitudes and relative phase, but not on the overall phase.

Example state shown above:

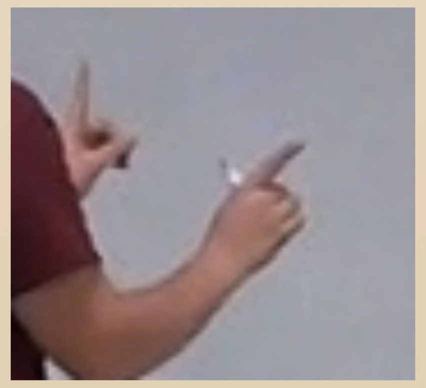
$$|\psi\rangle = \frac{1}{\sqrt{5}}\left(2e^{i\pi/4}|+z\rangle + 1e^{i3\pi/4}|-z\rangle\right)$$

## NPDs for states A, B, and E from the “Distinguishable states” problem




## The “Constant relative phase” (*CRP*) gesture

Match index fingers to the phase angles in the NPD for one state



Rotate both fingers through the same angle and match one finger to the phase angle of one of the components

Does the other finger match the phase angle of the other component?

***If so, the relative phase of the two states is the same.***

***If not, it’s not.***

**References**

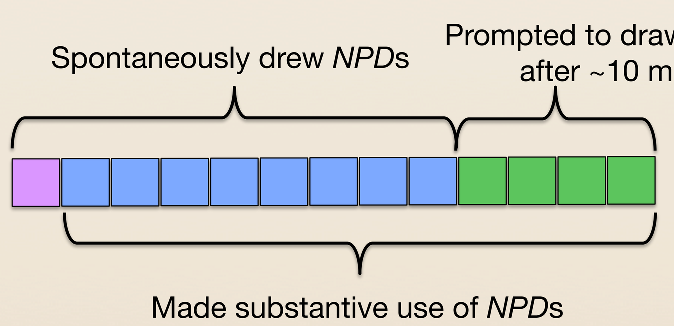
1. C. Singh, “Student understanding of quantum mechanics at the beginning of graduate instruction” *Am. J. Phys.* **76** (3), 277-287 (2008).
2. G. Zhu & C. Singh, “Surveying students’ understanding of quantum mechanics in one spatial dimension,” *Am. J. Phys.* **80** (3), 252-259 (2012).
3. G. Zhu & C. Singh, “Improving student’s understanding of quantum mechanics via the Stern-Gerlach experiment” *Am. J. Phys.* **79** (5), 499-507 (2008).
4. S. B. McKagan et al., “Developing and researching PhET simulations for teaching quantum mechanics” *Am. J. Phys.* **76** (4&5), 406-417 (2008).
5. H. G. Close, E. W. Close, & David Donnelly, “Nesting in graphical representations in physics,” PERC 2012 proceedings, 110-113.
6. E. R. Tufte, Beautiful Evidence, Graphics Press, Cheshire, CT (2006).
7. D. H. McIntyre, Quantum Mechanics, Pearson, Boston (2012).
8. G. Lakoff & M. Johnson, *Philosophy in the flesh: The embodied mind and its challenge to Western thought,* New York, Basic Books (1999).

## INTRODUCTION

Quantum mechanics is regarded by most who encounter it as extremely abstract and difficult to understand. It is also traditionally presented mostly in algebraic form with expressions that are difficult to interpret in geometric or physical terms. Studies on student difficulties in quantum mechanics show overall that students are often at a loss to explain algebraic expressions conceptually or graphically.[1,2] Research on student understanding of spin[3] indicates that students have difficulty distinguishing between physical 3-d “laboratory” space and the 2-d complex Hilbert space of spin quantum states. The instructional tools and findings presented in this article may contribute to solving this problem by helping students build geometric intuition about spin. This article is also part of a sustained research project investigating student learning of complex numbers and functions, both in terms of real and imaginary parts as well as in terms of phase. Previous research by others[1,2,4] has shown that many students have difficulty correctly applying and interpreting imaginary parts and phases.

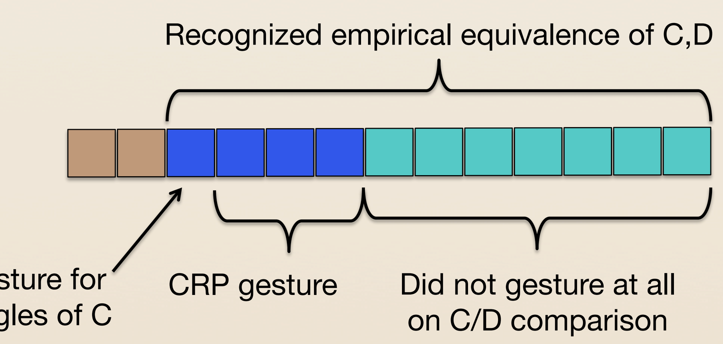
## RESULTS AND DISCUSSION

### Frequent, coherent use of *NPDs*

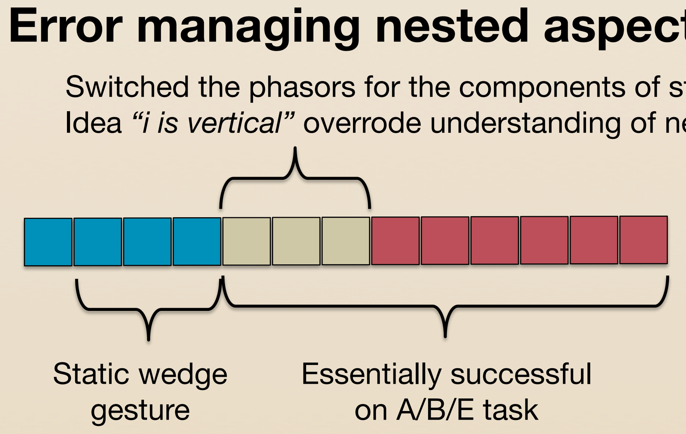


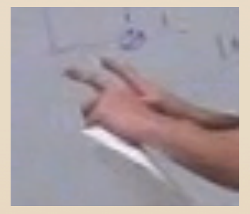

*N=13*

### Little gesturing for C/D




### Error managing nested aspect in A/B/E



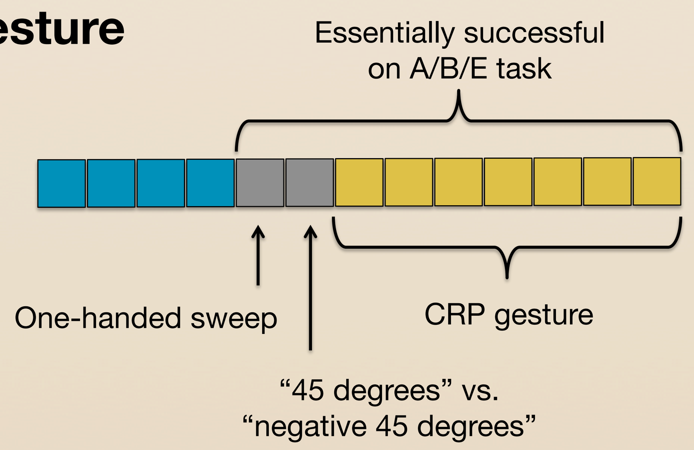



**Static wedge gesture in A/B/E**

*“Relative phase is just the distance between the phases”*

Speech and gesture offer coherent evidence of conceptual error.

### Success on A/B/E task mediated by CRP gesture




Both gray students and one gold student were holding a marker or paper that might have inhibited gesturing, but the gold student performed CRP while holding a marker *and* a paper.

## IMPLICATIONS

### Implications for instruction

Students have shown their ability to make productive use of a graphic tool, the *NPD*, to represent two complex coefficients for spin-1/2 quantum states, which are traditionally not conceived graphically or geometrically. Increased geometrization of canonical physics content can be of some advantage to students. We have also shown that students made productive use of their bodies (via *CRP*) in cognitive interaction with the graphic. Instruction should provide as many opportunities as possible for students to make sense of ideas, including the opportunity for body movement.

### Implications for research on cognition

Assessing relative phases by visual examination of the *NPD*s seems too difficult in the case of the A/B/E task, while performing a coordinated rotation of different fingers at different angles is apparently more manageable. Considering this fact together with students’ facility with *NPDs*, in which some differences between quantum states are conceived as rotation, makes us wonder what fundamental importance rigid-body rotation might have in cognition; in terms of both enacted rotation as an aid to imagined rotation, and imagined rotation as a metaphorical aid for understanding change in general.