

Students' Difficulties with Unit Vectors and Scalar Multiplication of a Vector

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Abstract. In this article we report an investigation on students' understanding of: 1) unit vectors and, 2) scalar multiplication of a vector. We administered two different tests to a total of 850 students after taking their first introductory physics course on mechanics at a large private Mexican university. The first test about unit vectors was taken by 270 students and the second test about scalar multiplication of a vector by 580 students. In the first part of this article, we analyze students' difficulties sketching the unit vector in the direction of a vector in the Cartesian coordinate plane. In the second part, we analyze students' responses in two types of problems: positive and negative scalar multiplication of a vector. In both parts of this article we describe frequent errors that have not been reported in the literature.

Keywords: unit vector, scalar multiplication of a vector, difficulties, misconceptions.

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INTRODUCTION

In recent years, researchers have investigated students' understanding of vector concepts [1-6]. However, few investigations have considered the understanding of unit vectors and scalar multiplication of a vector. In this article we focus on these concepts.

This article covers two objectives: 1) to analyze misconceptions with unit vectors and 2) to investigate difficulties with positive and negative scalar multiplication of a vector. In the first part of this article, we study students' difficulties sketching the unit vector in the direction of a vector in the Cartesian coordinate plane. Some research has focused on students' understanding of spherical unit vector, and described students' difficulties writing vectors in terms of spherical unit vectors [7]. In the second part, we investigate students' understanding of positive and negative scalar multiplication of a vector. Van Deventer [6] made an interview-based study to analyze difficulties with positive scalar multiplication of a vector. The particular feature of this part of our investigation is that we use open-ended problems to compare students' answers with positive and negative scalar multiplication of a vector.

In the following section we present the details of the methodology. Then, we divide the Results and Discussion section into three subsections. In the first two subsections we cover the two objectives of the study and in the third subsection we compare our results with other studies. At the end, we present a

Conclusions section with a review of the main results of the study.

METHODOLOGY

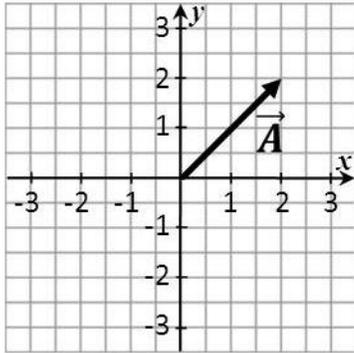
This research was conducted at a large private Mexican university. We used open-ended questions administered to hundreds of students having already taken the first calculus-based introductory physics (mechanics) course. Answers from students were analyzed and categorized.

Fig. 1 shows the problems used in this study. To address the first objective, we designed Problem 1. Its design was based in some preliminary results found in a previous research study. To address the second objective, we designed Problem 2 & 3. Problem 2 is a modified version of a problem used by Van Deventer [6]. We decided to include the coordinate axis because of a students' misconception reported by Van Deventer. Problem 3 is the same as Problem 2 but with a negative scalar. In all problems, students drew vectors in a provided empty grid.

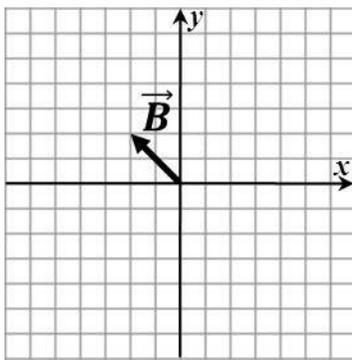
We administered Problem 1 to 270 students of a calculus-based physics course that consists of fluids, thermodynamics and waves. We also administered Problem 2 & 3 to 580 students of a calculus-based Electricity and Magnetism (EM) course. To make comparisons between Problem 2 & 3, we decided to divide the EM sample into two different groups (each of approximately 285 students), following the methodology used by Barniol and Zavala [8]. The selection was made randomly.

Unit Vector Problem:

Problem 1. A vector **A** is shown. Draw the unit vector in the direction of vector **A**.

**Scalar Multiplication of a Vector Problems:**

A vector **B** is shown.



Problem 2. Draw the vector **C**, if $C = 3B$.

Problem 3. Draw the vector **D**, if $D = -3B$.

FIGURE 1. Complete set of questions administered to students in the study. In all the problems students drew the vectors in an empty grid.

RESULTS AND DISCUSSION

This section is divided into three subsections. In the first two subsections we cover the two objectives of this study and in the third subsection we compare our results with other studies.

1. Difficulties with Unit Vectors

In this subsection, we analyze students' difficulties with unit vectors (objective 1). Table 1 shows the proportion of students' different representations of the unit vector in the direction of vector **A** as posed by Problem 1. Fig. 2 shows the most common representations drawn by students.

Table 1 shows that only 22% of students drew correctly the unit vector in Problem 1. Note that we considered as correct all the vectors in the direction of vector **A** with magnitudes between the magnitude of $0.5\hat{i} + 0.5\hat{j}$ vector and that of $\hat{i} + \hat{j}$ vector.

TABLE 1. Unit vectors sketched in Problem 1.

Unit Vector Sketched	%
Correct	22%
$\hat{i} + \hat{j}$ vector	25%
Two vectors: $2\hat{i}$ and $2\hat{j}$	14%
$0.5\hat{i} + 0.5\hat{j}$ vector	10%
Vector A	6%
$2\hat{i}$ vector	3%
No answer	4%
Others	16%

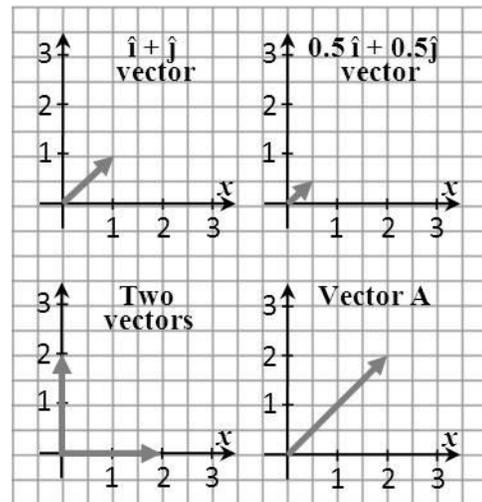


FIGURE 2. Most common representations in Problem 1.

As shown in Table 1, the most common error that students make in Problem 1 (25%) is to sketch a $\hat{i} + \hat{j}$ vector. These students don't have difficulties with the unit vector direction but have misconceptions with the magnitude of this vector and/or in the x - and y -components of this vector. Students usually sketched the $\hat{i} + \hat{j}$ vector directly without operations, so it is difficult to make a complete analysis of this representation. However, we found that some students wrote explanations as a part of their procedures that exemplify this error: "magnitude 1, in the same direction". This reasoning shows that some students understand that a unit vector is a vector which magnitude is one but they consider that the $\hat{i} + \hat{j}$ vector fulfill this requirement.

14% of the students sketched the unit vector of vector **A** as two vectors: $2\hat{i}$ and $2\hat{j}$ vectors. Approximately 70% of these students drew this two vectors in a tail-to-tail representation (as shown in fig. 2), and 30% of these students sketched them in a head-to-tail representation. A student, who made this error, reasoned as follows: "I consider that the unit vector is the sum of vectors indeed in i and j ". Another student

wrote the expression “ $\mathbf{A} = 2\hat{i} + 2\hat{j}$ ” as part of their procedures and labeled the two vectors as “ $\mathbf{A}_x = 2\hat{i}$ ” and “ $\mathbf{A}_y = 2\hat{j}$.” It seems that when these students are being asked about the unit vector in the direction of vector \mathbf{A} , they think on the unit-vector notation of vector \mathbf{A} ($\mathbf{A} = 2\hat{i} + 2\hat{j}$), and associate the unit vector in the direction of vector \mathbf{A} with the two components of vector \mathbf{A} written in the unit-vector notation ($2\hat{i}$ and $2\hat{j}$).

10% of students drew the unit vector in the direction of \mathbf{A} as a $0.5\hat{i} + 0.5\hat{j}$ vector. These students, as those who sketch the $\hat{i} + \hat{j}$ vector, don't have difficulties with the unit vector direction but have misconceptions with the magnitude of this vector and/or in the x - and y -components of this vector. These students usually sketched the $0.5\hat{i} + 0.5\hat{j}$ vector directly. However, we found that some of them made an error finding the unit vector with the definition: $\mathbf{A}/|\mathbf{A}|$. Students incorrectly established that the magnitude of vector \mathbf{A} is 4.

6% of students sketched vector \mathbf{A} as the unit vector in the direction of \mathbf{A} . Most of the students drew the vector directly. However, we found a student procedure that illustrates this error. The student drew first two vectors in a head-to-tail representation as the two vectors error in fig. 2. He labeled one vector as “ $2\hat{i}$ ” and the other vector as “ $2\hat{j}$ ”. Then he drew one dot line from the point (2, 0) to the point (2, 2) and another dot line from the point (0, 2) to the point (2, 2). Finally, he sketched the unit vector from point (0, 0) to point (2, 2), and labeled it as “ $\mathbf{u} = 2\hat{i} + 2\hat{j}$ ”. We see that at the beginning these students have the same reasoning as the ones that drew two vectors. They associate the unit vector in the direction of vector \mathbf{A} with the two vector components written in the unit-vector notation ($2\hat{i}$ and $2\hat{j}$). The difference is that students from this category state that the unit vector is the sum vector of these two vectors, which is vector \mathbf{A} .

As shown in Table 1, 3% of students sketched a $2\hat{i}$ vector. All of them sketched this vector directly, but probably they have a similar reasoning as the ones who drew two vectors. We also see that 4% of the students didn't answer the problem. Finally, 16% of students' answers didn't allow us to establish incorrect answers with significant proportion.

2. Difficulties with Scalar Multiplication of a Vector

In this subsection we investigate students' difficulties with positive and negative scalar multiplication of a vector (objective 2). Table 2 shows the proportion of students' different representations in drawing vector \mathbf{C} and vector \mathbf{D} in Problem 2 & 3. Fig. 3 shows the most common representations drawn by students in Problem 3.

TABLE 2. Vectors $\mathbf{C} = 3\mathbf{B}$ sketched in Problem 2 and vectors $\mathbf{D} = -3\mathbf{B}$ sketched in Problem 3.

Number of Problem and Responses	%
Problem 2 ($\mathbf{C} = 3\mathbf{B}$)	
Correct	96%
Others	4%
Problem 3 ($\mathbf{D} = -3\mathbf{B}$)	
Correct	74%
Incorrect magnitude, correct direction	7%
Perpendicular (clockwise), correct magnitude	5%
Perpendicular (counterclockwise), correct magnitude	4%
Vector \mathbf{B} being translated	3%
Opposite direction, correct magnitude	2%
Others	5%

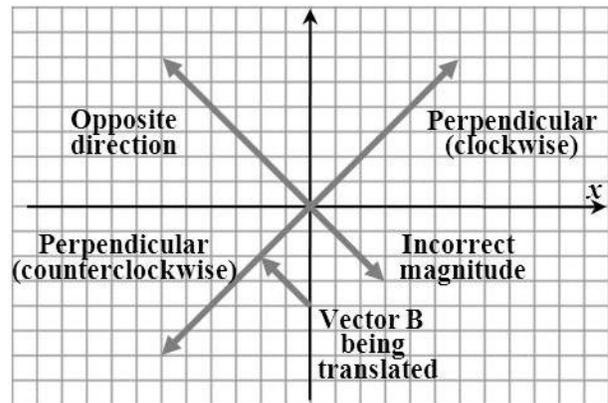


FIGURE 3. Most common representations in Problem 3.

Table 2 shows that a high fraction of students correctly sketched the vector $\mathbf{C} = 3\mathbf{B}$ in Problem 2. In this problem, we didn't find errors that could be grouped in a category with significant proportion.

On the other hand, Table 2 shows that a lower fraction of students (74%) correctly drew the vector $\mathbf{D} = -3\mathbf{B}$ in Problem 3. In this problem we found errors that could be grouped in categories. Comparing results between positive and negative scalar multiplication, we see that in the same population of students, a very low proportion (4%) have difficulties with the positive scalar multiplication, and a considerable fraction (26%) have difficulties with the negative scalar multiplication.

According to Table 2, 7% of students who solved Problem 3 sketched a vector \mathbf{D} with a correct direction but incorrect magnitude. We found two common incorrect magnitudes. If we consider that each square represents one unit, the first error is to sketch a $3\hat{i} - 3\hat{j}$ vector (see fig. 3) and another error is to draw a $2\hat{i} - 2\hat{j}$ vector. It seems that these students have difficulties

interpreting what role the real number 3 plays in the multiplication.

5% of students sketched a vector \mathbf{D} perpendicular to vector \mathbf{B} (clockwise rotation of 90° from vector \mathbf{B}) with correct magnitude (see Table 2). Approximately half of these students sketched vector \mathbf{D} directly. The other half of students made a sign error in the unit-vector notation. They stated that vector “ $\mathbf{B} = -2\mathbf{i} - 2\mathbf{j}$ ” and therefore calculated that vector “ $\mathbf{D} = 6\mathbf{i} + 6\mathbf{j}$.”

4% of the students made a similar error. These students sketched a vector \mathbf{D} perpendicular to vector \mathbf{B} with correct magnitude, but the vector is rotated counterclockwise from vector \mathbf{B} . In this case, all of these students drew vector \mathbf{D} directly without calculations. It seems that these students and some of them from the previous category have difficulties with the negative sign of the multiplication, and believe that $-\mathbf{B}$ is perpendicular to vector \mathbf{B} . However, these students don't have problems interpreting the role of the real number (3) on the scalar multiplication; they sketch vectors with correct magnitudes.

3% of the students translated vector \mathbf{B} three grid squares. Approximately half of these students translated vector \mathbf{B} three negative units in the y -axes (see fig. 3) as detected by Van Deventer [6]. We also found that some other students translated it three negative units in the x -axis and three negative units in the x and y axis. These students have difficulties not only with the interpretation of the negative sign, but also, with the role of the real number in a scalar multiplication.

Finally, 2% of the students sketched a vector with the correct magnitude but opposite direction. It seems that these students don't consider the negative sign of the multiplication.

3. Comparison with Other Studies

In this subsection we compare our results with other studies. In the first part of our research we study students' difficulties with Cartesian unit vectors. A research has described students' difficulties writing vectors in terms of spherical unit vectors [7]. The different focus of that research and the one presented here makes it difficult to compare.

Our results from Problems 2 & 3 are compared to Van Deventer's study [6]. That study consisted on interviews using physical and no-physical context problems. One error was found: one student translated the vector in a positive scalar multiplication problem in the physical context of acceleration. From these results, a no-context multiple-choice question was built. The distractors consisted on vectors being translated and vectors with correct direction but incorrect magnitudes. The test was administered to

students finishing a mechanics course and 95% of the students answered correctly and no student chose the options with vectors being translated. The most common error was to choose a vector with incorrect magnitude. In our study we also found that no student translated the vector with the positive scalar multiplication problem (Problem 2). However, we did find this error and other students' difficulties in the negative scalar multiplication of a vector.

CONCLUSIONS

We found that students who completed a calculus-based type course of mechanics have serious difficulties sketching the unit vector in the direction of a vector in the Cartesian coordinate plane. Only, 22% of the students answered correctly Problem 1. We can establish two general errors: 1) Misconception in the magnitude of the unit vector and/or in the x - and y -components of this vector, 2) Confusion between the unit vector in the direction of vector \mathbf{A} with the two component vectors of vector \mathbf{A} written in the unit-vector notation. We also found that students who completed a course of electricity and magnetism (last introductory course in this institution) still have difficulties with the negative scalar multiplication of a vector.

Even after a calculus-based course of mechanics, students still have difficulties with unit vectors and negative scalar multiplication of a vector. Instructors should be aware of these specific difficulties to plan the instruction.

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