

Johnson Noise and Shot Noise

Phys 1560 – Brown University – August 2010

Barus and Holley Room 203

Purpose

- Investigate and quantify Johnson noise and shot noise
- Use acquired data to measure Boltzmann's constant and the charge on an electron.



Introduction to Electrical Noise

Noise, as its name suggests, refers to unwanted signals encountered during data acquisition. Since acquisition of data always involves some amount of noise it is important to know what the noise is and how to reduce and quantify its appearance in final measurements. Electrical noise that comes from sources external to the apparatus can usually be eliminated by shielding or grounding. However, noise intrinsic to the equipment will remain. It is this intrinsic noise that we are going to study. A common type of noise is “one over f noise” ($1/f$). The strength of $1/f$ noise varies inversely with frequency; this noise is only problematic at lower frequencies. Noise whose strength is independent of frequency is called “white noise.” A Fourier transform of white noise gives a flat power spectrum.

You will be investigating two forms of intrinsic noise in electrical systems: Johnson noise and Shot noise. Johnson noise results from the thermal motion of electrons inside a conductor. Shot noise occurs in low-current systems where the number of charge carriers is small enough for statistical fluctuations in current to be detectable. At higher currents where random fluctuations in current are several orders of magnitude smaller than the main current shot noise will be washed out and not detected. Both Johnson noise and shot noise are forms of white noise.

Johnson Noise

Background

Electrical noise in resistors was measured by Johnson and theoretically described by Nyquist in 1928. In its differential form, the Nyquist formula says that for an infinitesimally small frequency interval, $d\nu$ (in Hertz), the mean square voltage across resistor R (in Ohms) at a temperature T (in Kelvin) is given

$$dV_J^2 = 4k_B TR d\nu \quad (1)$$

where Boltzmann's constant, $k_B = 1.38066 \times 10^{-23}$ J/K. An equation for the mean square voltage can then be obtained by integrating equation (1) over a range of frequencies. The brackets denote a time averaged quantity.

$$\langle V^2 \rangle = 4k_B TR \Delta\nu \quad (2)$$

One might naively expect that selecting an infinite frequency range would yield an infinite mean square voltage across the resistor; this does not happen. Equations 1 and 2 are Nyquist's low frequency approximations, valid whenever $hf \ll k_B T$. This is analogous to the relationship between the Rayleigh-Jeans formula (ultraviolet catastrophe) and Planck's law of black-body radiation. Refer to the references for a more detailed theoretical discussion.

Apparatus

You will need the following equipment: digital oscilloscope, low noise preamplifier, function generator, attenuator, bud box (metal box which shields external noise,) collection of resistors, and a thermometer. Figure 1 is a block diagram of the measurement system. Note: all wires are coaxial cables.

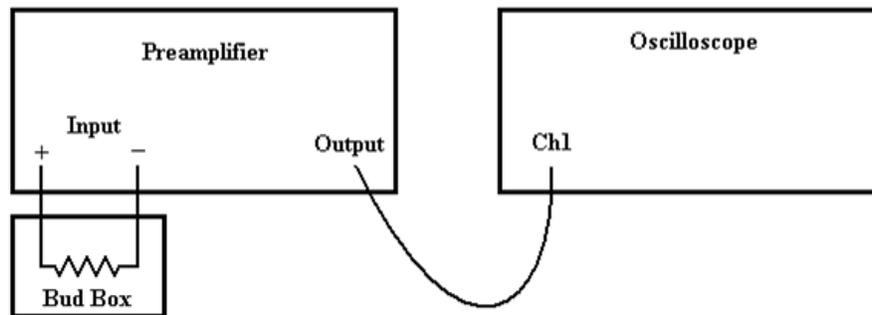


Figure 1. Basic schematic diagram for measuring Johnson noise.

Effective Circuit Characteristics

Consider the impact of the measurement system. Johnson noise is too small to measure directly with an oscilloscope. To amplify the signal to a measurable level it is sent through a preamplifier. Unfortunately, the preamplifier is not ideal and has its own noise characteristics that must be taken into account.

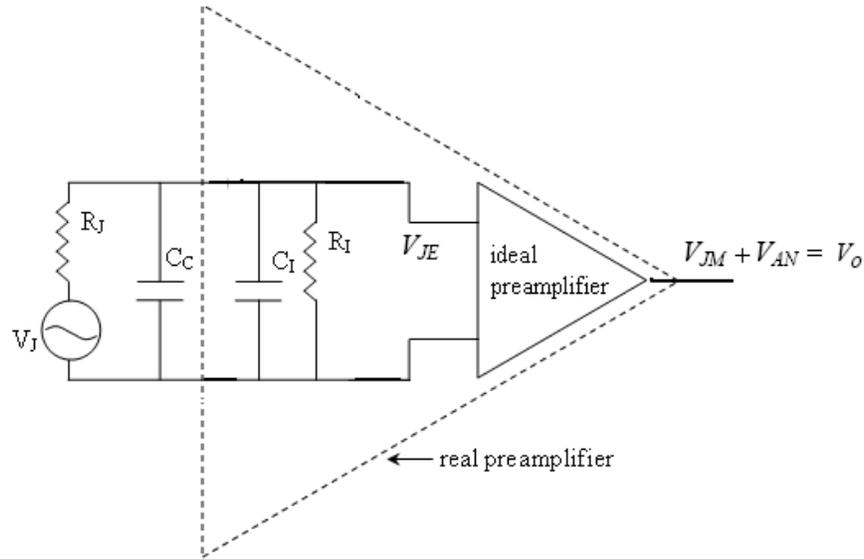


Figure 2. Circuit details of resistor R_J & preamp from figure 1 (Kittel et al).

Definitions for Figure 2:

R_J = Resistance being measured

V_J = Johnson noise generated by R_J

C_c = Capacitance of coaxial cable

R_I = Input resistance of the preamplifier

C_I = Input capacitance of the preamplifier

V_{JE} = Johnson noise seen by input of the preamplifier

V_{JM} = Johnson noise measured at the output of the preamplifier

V_{AN} = Noise generated by the preamplifier

V_O = Total output voltage of the preamplifier

The coaxial cable connecting the resistor to the preamp has capacitance (C_c) and the preamplifier input has capacitance (C_I). Since they are in parallel their total capacitance can be reduced to $C_T = C_c + C_I$. This “stray capacitance” shorts (shunts) a portion of the signal to ground thus reducing the Johnson noise seen by the ideal preamplifier to V_{JE} .

$$V_{JE}^2 = \frac{1}{1 + (\omega C_T R_J)^2} V_J^2 \quad (3)$$

Equation 3 shows that V_{JE} depends on the angular frequency ω (thus the measured voltage V_{JM} will not be white noise). See Appendix A for the derivation.

Preamplifier Gain

To calculate Johnson noise V_J from the preamp output V_O the pre amplifier gain as a function of frequency must be determined.

Suppose a signal of amplitude V_{in} with frequency ν (known as a Fourier mode) enters an ideal preamplifier, its output amplitude V_{out} will either increase or decrease. The gain function can be defined by

$$g(\nu) = \frac{V_{out}(\nu)}{V_{in}(\nu)} \quad (4)$$

The preamplifier has adjustable filters and gain. Figure 3 shows how a graph of how $g(\nu)$ might appear with a band-pass filter.

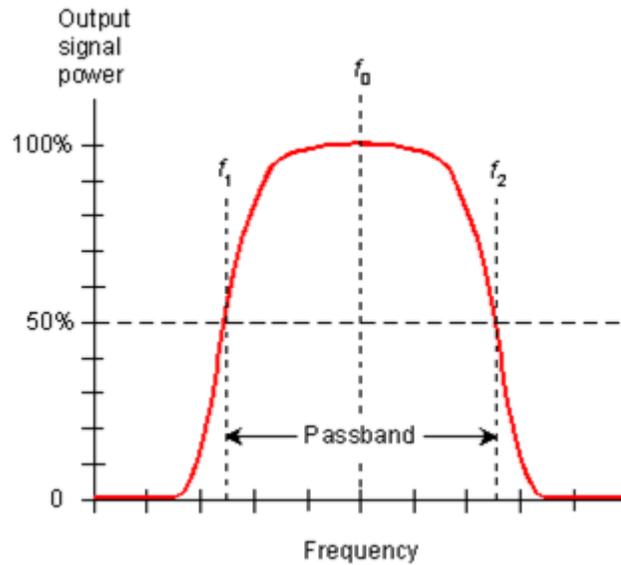


Figure 3. Plot of gain versus frequency for a band-pass filter: frequencies well below the cutoff receive maximum gain; gain varies sharply with frequency near the cutoff. (Courtesy: Jakob Ashtar & SearchCIO-Midmarket.com)

Johnson noise is composed of an infinite number of Fourier modes. Ignoring preamplifier noise for each mode we have

$$dV_{JM}^2 = g^2(\nu) dV_{JE}^2 \quad (5)$$

Integrating over all frequencies

$$\langle V_{JM}^2 \rangle = \int_0^\infty \frac{d\langle V_{JE}^2 \rangle}{d\nu} g^2(\nu) d\nu \quad (6)$$

Since Johnson noise is “white,”

$$\frac{d\langle V_{JE}^2 \rangle}{d\nu} = \frac{4k_B TR_J}{1 + (\omega C_T R_J)^2} \quad (7)$$

so equation 6 can be rewritten as

$$\langle V_{JM}^2 \rangle = 4k_B T R_j \int_0^\infty \frac{g^2(v)}{1 + (\omega C_T R_j)^2} dv \quad (8)$$

Define the “effective gain” $G(v)$ as

$$G(v) = \frac{g^2(v)}{1 + (\omega C_T R_j)^2} \quad (9)$$

Now we can rewrite $\langle V_{JM}^2 \rangle$ as

$$\langle V_{JM}^2 \rangle = 4k_B T R_j \int_0^\infty G(v) dv \quad (10)$$

Note: you will measure $G(v)$, not $g(v)$.

Preamplifier Noise

Now consider the effects of preamplifier noise V_{AN} : Let $V_o = V_{JM} + V_{AN}$ and V_{JM} & V_{AN} are independent voltage sources, $\langle V_o^2 \rangle$ can be written:

$$\langle V_o^2 \rangle = \langle (V_{JM} + V_{AN})^2 \rangle = \langle V_{JM}^2 \rangle + \langle V_{AN}^2 \rangle + 2\langle V_{JM} \rangle \langle V_{AN} \rangle \quad (11)$$

The last term on the right side vanishes because both V_{JM} and V_{AN} average to zero. Therefore V_{JM} , V_{AN} , and V_o add in quadrature.

$$\langle V_{JM}^2 \rangle = \langle V_o^2 \rangle - \langle V_{AN}^2 \rangle \quad (12)$$

Where $\langle V_{AN}^2 \rangle$ is found by shorting the preamplifier’s input.

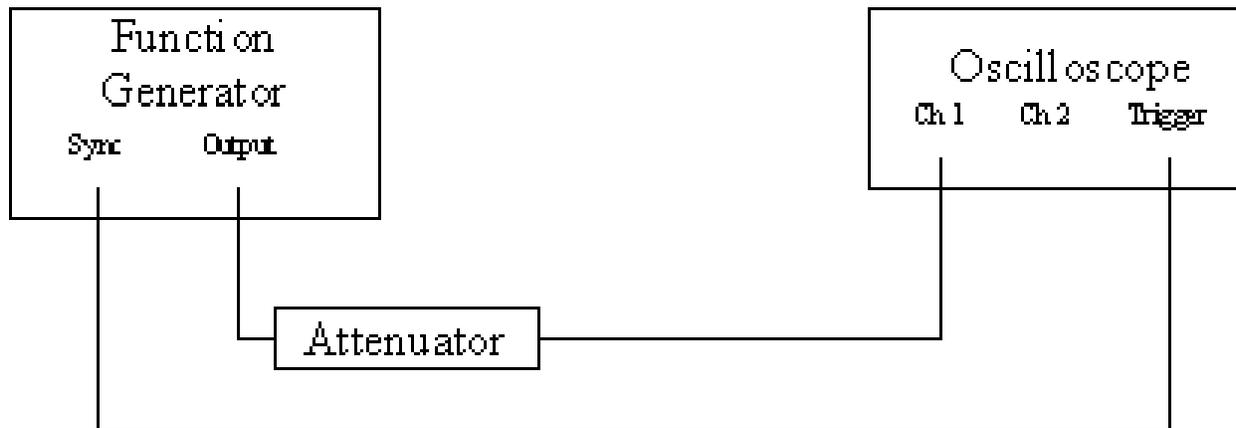


Figure 4a. Circuit diagram for measuring V_{in} .

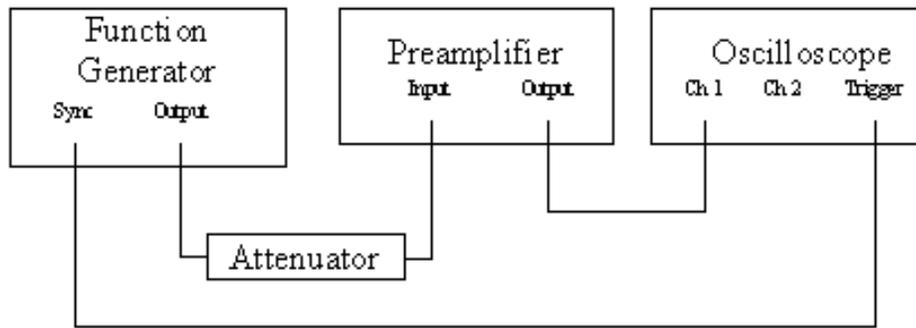


Figure 4b. Circuit diagram for measuring V_{out} .

Measuring Effective Gain “ $G(v)$ ”

- 1) Perform the following measurements using the setups shown in figures 4a & 4b. Once with the preamp filters set to 3 kHz and 1 kHz, then again with them set to 30 kHz and 10 kHz. Note: To compute the integral of equation 11, the range of frequencies measured must be much wider than the frequency range set on the preamp (must allow $G \rightarrow 0$ as $v \rightarrow 0$ or ∞). Note, unless otherwise stated, always use a sine wave.
- 2) Finding the appropriate gain setting for the preamplifier. While it is advantageous to maximize the gain in order to get the best possible reading of the Johnson noise itself, you’ll also need to get accurate measurements of the effective gain in order to extract the Boltzmann constant. To do this, use setup 4b and apply the center frequency of the filter’s bandwidth. Adjust the attenuation on the input signal to the minimum necessary to prevent an overload in the preamplifier. Switch to setup 4a and observe the signal on the oscilloscope. If the observed sine wave is not easily discernable or is seriously distorted by noise, this gain setting will not provide usable data. Find the highest gain setting that provides a discernable and undistorted sine wave with setup 4a. Assemble the circuit in figure 1 using the largest resistor to be measured. Since Johnson noise is proportional to resistance, this will be the maximum noise you will measure. Make sure the gain setting you just selected does not overload the preamplifier. Remember, to get good data in this experiment you need to maximizing the amplification while not getting a distorted signal out of the amplifier. And of course, do not overloading the preamp input. Note, once the gain for a given bandwidth, has been determined do not change it.
- 3) Select a frequency on the function generator.
- 4) Using setup 4b, apply the minimum possible voltage to the attenuator. Adjust the attenuator to the maximum possible voltage that does not over load the preamplifier. Measure the preamp’s RMS output voltage on the scope.
- 5) Using the same cables and without changing any setting on the function generator or attenuator, switch to setup 4a. Measure the preamp input voltage (RMS) used in setup 4b.
- 6) Repeat steps 3 - 5 for a number of frequencies. To determine a reasonable frequency range and the appropriate frequency intervals, consider that the noise applied to the filter spans the entire spectrum not just frequencies somewhat larger and smaller than the filter’s bandwidth.
- 7) Repeat this procedure for the other preamp filter band.

Measuring Johnson Noise

- 1) Become familiar with the operation of the oscilloscope. Learn to measure voltage with the cursors, display and measure V_{RMS} (check the “measurements menu,) signal average, and zoom. The preamplifier can be powered by the 120V AC line or by an internal battery. V_{AN} is significantly higher when the preamplifier is powered by line voltage. NOTE: The oscilloscope is not ideal and is a source of error. To measure the error of the oscilloscope, record multiple values of the displayed RMS voltage. Find the mean and standard deviations. Check all time/division settings for any periodic noise.
- 2) Set up the measurement circuit shown in figure 1. Use the shortest cables possible to reduce noise. Use the same cables for all measurements. Set the preamp low-pass filter to 3 kHz, high-pass filter to 1 kHz and coupling to DC (Direct Coupling). Keep the same gain settings used in the Measuring Effective Gain section.
- 3) Beware of ground loops. It is easy to inadvertently create a ground loop with the BNC cable shields. As you may already know, typical circuit convention uses red for positive, black for negative, and green for ground. On devices with BNC terminals, it is common for the shield of the BNC to be connected to negative and ground. BNC cables contain an internal conducting wire (signal), surrounded by an insulating layer which is surrounded by an external conducting mesh (shield). When this mesh is connected to ground, it shields the internal conductor from noise. In a BNC to banana adapter, the shield of the BNC connects to the negative side of the banana. The negative side of a banana adapter can be identified by a small plastic nub on the side. It is easy to create ground loops by connecting the signal of a BNC to ground when using adapters. Make sure to check the polarity of all the adapters you use. Beware, all grounds by definition are connected to each other but not all BNC shields are grounds and therefore have a voltage relative to ground. This voltage will be measured by the setup and will add error to your measurements. Be sure to understand the ground and shield connections in this setup.
- 4) Record *all* relevant apparatus settings then measure the RMS voltage $\langle V_0^2 \rangle^{1/2}$ of all resistors provided.
- 5) Find V_{AN} - the noise generated by the preamp – by shorting the preamp input to ground via coupling and measuring the RMS voltage at its output. Do not change any other preamp setting.
- 6) Repeat steps 4 - 6 with the low-pass filter set to 30 kHz and high-pass filter set to 10 kHz.
- 7) In order to ensure your data is roughly correct, you can make an approximation of the integral of the gain in equation 10 by treating the filters as ideal, i.e. by setting the value of the integral as the bandwidth of the filter times the value of the gain squared. This will not give you an accurate calculation of the Boltzmann constant, but will make sure you are on the correct order of magnitude.

Measuring Johnson Noise with Temperature Variation

- 1) Revert to the equipment configuration shown in figure 1 but instead of the bud box use the resistor shielded with grounding braid. Measure the resistance of this special resistor with a multimeter.

- 2) Use the steps described under “*Measuring Johnson Noise*” to measure the Johnson noise of this resistor under the following conditions. Be sure the resistor has reached thermal equilibrium before taking measurements and be sure to record the temperature. Also make sure to perform the measurements in the following order, as each successive bath will evaporate the previous cryogen. Note: put the boiling water in a separate container, do not perform the measurement in the kettle, as the elevated temperature of the kettle’s heating element, along with the physical motion caused by the boiling, generate extra noise. However, you will need to perform the measurement quickly, before the water cools.
- Room Temperature
 - Liquid Nitrogen
 - Dry ice in alcohol
 - Boiling Water

Analysis

Analyze your data using the equations provided above. Plot-fit the result to recover Boltzmann’s constant ($k_B = 1.3806504(24) \times 10^{-23}$ J/K). Include error bars on both axes. Disregard data affected by shunt capacitance.

Use equation 12 to correct for preamplifier noise before considering the effective gain. The effective gain $G(\nu)$ is a frequency dependent variable, however, the integral of $G(\nu)$ is a constant - see equation 11. You may have a hard time making sense of a plot of $G(\nu)$. Try making a plot of $G(\log_{10}(\nu))$ and fitting that to a Gaussian. This will simplify the process of evaluating $\int_0^\infty G(\nu) d\nu$, but don’t forget the Jacobian!! Also, make sure your units all match—convert mV to V and kHz to Hz.

Shot Noise

Introduction

Shot noise was described by Schottky in 1918. He drew an analogy between the noise created by statistical fluctuations in current and the fluctuating noise heard when lead shot hits a hard surface. Hence this phenomenon was dubbed “shot noise.” Look up “shot towers” online for a further historical explanation of the analogy.

Shot noise is caused by rate fluctuations of independent charge carriers and can only be detected at very low currents where charge carrier motion is statistically independent. At higher currents, where charge carriers are not statistically independent, shot noise is washed out. Current variations associated with shot noise can be described with Poisson statistics. Schottky showed the mean square current fluctuation is:

$$\langle \Delta I^2 \rangle = 2eI_{ave} \Delta f \text{ where } \Delta I = I_{inst} - I_{ave} \quad (13)$$

Since shot noise is white noise these current fluctuations can be related to voltage fluctuations via Ohms Law. Therefore, the mean square voltage fluctuation across resistor R is:

$$\langle V_s^2 \rangle = \langle V_{tot}^2 \rangle - \langle V_{amp}^2 \rangle = 2eRV_{s,ave} \int_0^\infty g^2(\nu) d\nu \quad (14)$$

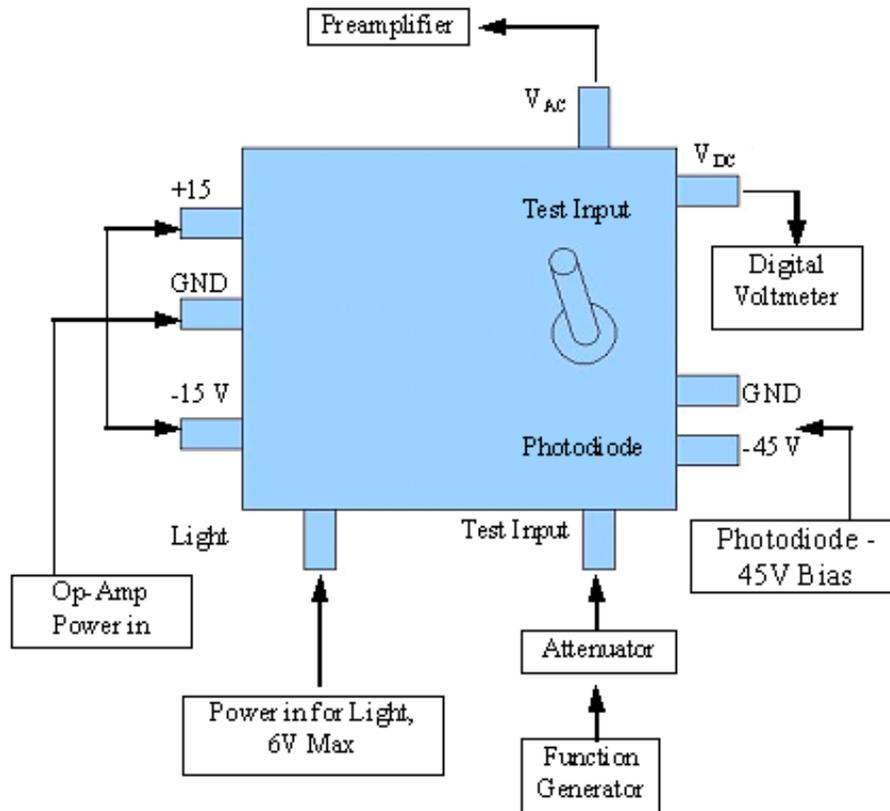


Figure 7. Shot Box Connection Diagram

Procedure

- 1) Set up the circuit shown in figure 7. Set the preamplifier as follows: coupling to AC, gain to 1×10 , low-pass filter to 10 kHz and high-pass filters to 1 kHz. Make sure to turn on the power supply to the op-amp.
- 2) Determine the measurement system's frequency dependent effective gain by setting the Shot Box switch to "Test Signal" and following the procedure outlined in "Measuring Effective Gain" section of Johnson noise.
- 3) Remove the test signal from the "Test Input", set the Shot Box switch to "Photodiode" and apply -45V bias to the photodiode. Vary the voltage to the light – do not to exceed 6V - measure the corresponding DC voltage, $V_{s,AVE}$ and RMS voltage, $(\Delta V_{tot}^2)^{1/2}$.
- 4) Determine the noise generated by the measurement system, $\langle V_{amp} \rangle$, by setting the Shot Box switch to "Test Signal", shorting the "Test Input" to ground and measuring the "AC Signal Out" RMS voltage. Do not change any pertinent apparatus settings.

Analysis

Use the technique described in Johnson noise to analyze your data and find the charge on the electron.

References

- F. Reif, *Fundamentals of Statistical and Thermal Physics*, pages 582 through 594.
 Kittel and Kroemer, *Thermal Physics*, pages 98 through 102.
 J. Johnson, "Thermal Agitation of Electricity in Conductors", *Phys. Rev.* **32**, 97 (1928)
 H. Nyquist, "Thermal Agitation of Electric Charge in Conductors", *Phys. Rev.* **32**, 110 (1928).
 J. L. Lawson and G. E. Uhlenbech, *Threshold Signals*, (Dover, New York, 1965).
 J. A. Earl, "Undergraduate Experiment on Thermal and Shot Noise", *Am. J. Phys.* **34**, 575 (1966).
 P. Kittel, W. R. Hackleman, and R. J. Donnelly, "Undergraduate experiment on noise thermometry," *Am. J. Phys.* **46**, 94 (1978)

APPENDIX A

Ohm's law states, $V = IR$ where R represents the medium's opposition to current flow. In DC circuit theory R is the only quantity that inhibits current flow.

In AC circuit theory opposition to current flow is determined by impedance Z . Impedance consists of DC resistance R and reactance of the capacitive and inductive elements in the circuit. As opposed to resistance reactance is the frequency dependent and causes phase changes.

Impedance is a complex number consisting of real (resistance) and imaginary (reactance) components. Impedance represents the total opposition to current flow in the circuit. We write impedance as $Z = |Z|e^{i\theta}$, where $|Z|$ is the magnitude of the impedance and θ represents the phase shift. Both AC voltage and AC current are written as complex numbers denoted by \vec{V} and \vec{I} . Therefore, the complex version of Ohm's law is $\vec{V} = \vec{I}Z$. In a pure capacitor with capacitance C or inductor with inductance L , the reactance X is equal to the impedance Z . In a pure resistor with resistance R , the resistance R equals the impedance Z . See Below:

$$X_C = Z_C = \frac{1}{i\omega C} \quad X_L = Z_L = i\omega L \quad Z_R = R$$

where ω is the angular frequency of the voltage source.

To combine impedances in parallel use $Z_{tot} = \frac{Z_1 Z_2}{Z_1 + Z_2}$

To combine impedances in series use $Z_{tot} = Z_1 + Z_2$

Example:

To determine a relationship between the voltage measured and the actual Johnson noise voltage, consider figure 8 below.

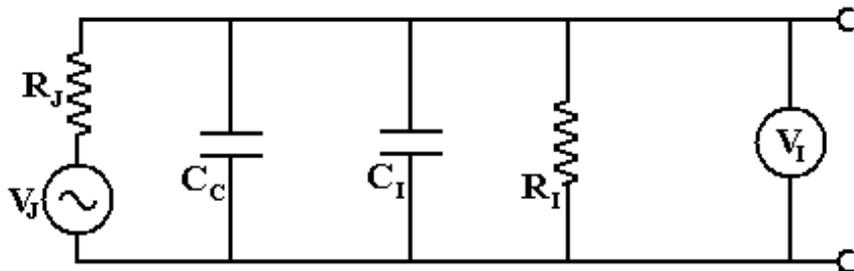


Figure 8. Expansion of input part of figure 2. Johnson noise resistor = $R_J + V_J$, capacitance of coaxial cable = C_C , input capacitance of preamp = C_I , and input impedance of preamp = R_I

Note, this is a parallel circuit where $C_T = C_1 + C_2$. Since C_T is parallel to R_I their total combined impedance is

$$Z_{C_T R_I} = \frac{\frac{-i}{\omega C_T} R_I}{\frac{-i}{\omega C_T} + R_I} \quad (1)$$

From Ohm's law for AC circuit, we have the following expression

$$\tilde{V}_I = \frac{Z_{C_T R_I}}{R_J + Z_{C_T R_I}} \tilde{V}_J = \frac{R_I}{R_I + R_J + i\omega C_T R_I R_J} \tilde{V}_J \quad (2)$$

Of course the voltage you will measure will not be complex; the imaginary part of equation 2 indicates that there is a phase shift between the voltage V_J and voltage V_I . We are not concerned with the phase shift so the amplitude of V_I is:

$$V_I = \frac{R_I}{\left[(R_I + R_J)^2 + (\omega C_T R_I R_J)^2 \right]^{1/2}} V_J \quad (3)$$

By factoring R_I out of the denominator and noting that $R_J \ll R_I$, equation 3 can be approximated:

$$V_I^2 = \frac{1}{1 + (\omega C_T R_J)^2} V_J^2 \quad (4)$$