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The effect of a soap film on a catenary: measurement of surface tension from the triangular configuration

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Abstract
A chain assumes the well-known shape known as a catenary when it hangs loosely from two points in a gravitational field. The correct solution of the catenary was one of the early triumphs of the newly invented calculus of variations at the end of the 17th century. Here we revisit the catenary and show that, for a chain hanging from a horizontal rod, three new and distinct configurations are possible if a soap film covers the area bounded by the chain and the rod. We first review the general problem and discuss the conditions under which the chain assumes a concave, triangular or convex configuration. The deciding factor is the strength of surface tension relative to the gravitational force per unit length of the chain. The conditions under which the chain assumes the shape of a perfect triangle are discussed in greater detail and analysed to obtain the tension along the chain. The triangular configuration is especially intriguing to undergraduates and may be used as a simple experiment to obtain the surface tension of the soap solution by measuring just one angle of the triangle.

(Some figures in this article are in colour only in the electronic version)

1. Introduction
A chain assumes the familiar form known as a catenary when it hangs loosely from two fixed points in the gravitational field. The catenary has had a venerable place in the history of mathematics. Galileo believed the shape to be a parabola [1], which was later shown to be incorrect. Late in the 17th century the correct solution of the catenary problem was an early triumph of the nascent field of the calculus of variations. In fact during 1690–92, the correct solution was derived and published independently by several well-known mathematicians of the time including Leibnitz, Huygens and the two Bernoulli brothers Jacob and Johann [2].
In 1744, Euler recast the catenary as an isoperimetric problem of finding the equation of a hanging chain of fixed length that has the lowest centre of gravity. This, in essence, is the modern version of the catenary problem where the length constraint is incorporated with the introduction of a Lagrange multiplier and the solution sought is the configuration that minimizes the gravitational potential energy of the chain [3, 4].

What new configurations beyond the catenary are possible if, in addition to gravity, we introduce a second force, surface tension? More specifically, what shapes are possible when the chain hangs from a horizontal rod and a soap film covers the area bounded by the chain and the rod? The answer turns out to have a few delightful surprises.

In this paper, we first review the general problem and discuss the conditions under which the chain assumes a concave, triangular or convex configuration. The deciding factor turns out to be the relative strength of surface tension to the gravitational force per unit length of the chain. The conditions under which the chain assumes a perfect triangular shape are discussed in greater detail and analysed to obtain the tension in the chain. The triangular configuration is especially intriguing to undergraduates and lends itself to a simple experiment for measuring the surface tension of the soap solution.

2. The new configurations

To be more specific, consider a hanging chain of linear mass density $\lambda$ covered by a soap film of surface tension $\sigma$. To simplify the analysis, we neglect the mass of the soap film and assume the surface tension to be constant along the chain. The forces acting on an element of the chain, $ds$, are shown schematically in figure 1. In addition to the two tensions $T_1$ and $T_2$, the gravitational force on the element is $\lambda g ds$. The surface tension force pulling the element inwards is $2\sigma ds$, where the factor of 2 accounts for the fact that the soap film has two surfaces. Clearly the element $ds$ is in equilibrium under the combined action of these four forces.

Note that the surface tension force per unit length, $2\sigma$, is constant and acts normal to the chain pulling it inwards. On the other hand, the normal component of the gravitational force opposes the surface tension and tends to pull the chain outwards. In general, subject to some geometrical constraints, the strength of surface tension relative to the gravitational force determines the equilibrium configuration of the chain.
Figure 2. When the normal component of the gravitational force per unit length (\(\lambda g \cos \theta\)) exactly balances the force per unit length due to surface tension (2\(\sigma\)), the chain assumes a perfect triangular shape.

Figure 3. When the normal component of the gravitational force per unit length (\(\lambda g \cos \theta\)) is greater than the surface tension (2\(\sigma\)), the chain is pulled outwards and assumes a concave configuration.

How does geometry come into play? The component of the gravitational force acting normal to the chain depends on the local angle the chain makes with the horizontal. Referring to figure 1, when the distance 2\(X\) is relatively small, the chain hangs down and allows the surface tension to dominate the opposing normal component of the gravitational force all along the chain. In this case, the chain is pulled into a concave configuration as shown in figure 1.

As the distance 2\(X\) between the two support points is increased, the normal component of the gravitational force per unit length is increased until it exactly matches the strength of the surface tension force. Under this condition the chain assumes a perfect triangular shape as shown in figure 2.

When the distance 2\(X\) is increased further, the normal component of the gravitational force per unit length exceeds the surface tension. The chain is pulled outwards to assume a convex configuration as shown in figure 3.

3. Equilibrium conditions

The general equations which describe the equilibrium of an element of the chain are easy to write down. Referring to figure 3, the equilibrium of forces on an element \(ds\) along the chain is given by

\[
dT = \lambda g \sin \theta \, ds,
\]

(1)
where
\[ dT = T_2 - T_1. \] (2)
The equilibrium of forces normal to the element gives
\[ T \, d\theta + 2\sigma \, ds = \lambda g \cos \theta \, ds. \] (3)

Equations (1) and (3) are the two coupled differential equations whose solutions give the chain equation in terms of the parameter \( \lambda \) and the initial values chosen for \( X \) and \( L \). It is worth noting that these two differential equations apply equally well to the three configurations if we take into account the fact that \( (ds/d\theta) < 0 \) for the concave configuration of figure 1, whereas \( (ds/d\theta) = 0 \) for the triangular configuration of figure 2, but \( (ds/d\theta) > 0 \) for the convex configuration of figure 3.

Before considering the special case of the triangular configuration in detail, it is useful to sketch the general procedure for solving the coupled differential equations.

First, the differential equations may be decoupled by dividing equation (1) by equation (3):
\[ \frac{dT}{T} = \frac{d\theta(\sin \theta)}{(\cos \theta - \alpha)}. \] (4)
where \( \alpha \equiv (2\sigma/\lambda g) \) is a dimensionless constant.
Equation (4) can now be integrated at once to give tension \( T \) as a function of \( \theta \):
\[ T = \lambda g \frac{C}{(\cos \theta - \alpha)}, \] (5)
where \( C \) is a constant to be determined from the initial conditions.

We can now use equation (5) to substitute for \( T \) in equation (3) to get a decoupled differential equation,
\[ \frac{ds}{d\theta} = \frac{C}{(\cos \theta - \alpha)^2}. \] (6)

As expected, the detailed solutions of equation (6) are quite involved and depend on the choice of the initial conditions [5–7]. The triangular configuration, however, is exceptionally simple to analyse and provides an excellent opportunity to introduce the subject to undergraduates. It may also be used in a simple experiment to measure the surface tension of a soap solution by measuring just one angle.

4. Triangular configuration

In figure 2, the forces acting on an element of the chain \( ds \) are shown for the triangular configuration. An important characteristic of this configuration is the fact that \( \theta \) is a constant which leads to \( d\theta = 0 \). This condition in turn decouples equation (1) from equation (2) and simplifies the analysis considerably.

Referring to figure 2, we find that the equilibrium condition on \( ds \) along the chain requires
\[ dT = \lambda g \sin \theta \, ds. \] (7)
Note that equation (7) is identical to equation (1). However, the equilibrium of forces normal to the element yields
\[ 2\sigma \, ds = \lambda g \cos \theta \, ds. \] (8)
Since these two equations are naturally decoupled, equation (7) can be integrated at once to yield
\[ T = \lambda g \sin \theta s, \] (9)
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Figure 4. Picture of the chain showing the concave configuration.

where \( s \) is the distance from the chain tip, and equation (8) may be written as

\[
2\sigma = \lambda g \cos \theta.
\]  

Equation (9) at once reveals that the tension in the chain is zero at the lower tip where \( s = 0 \) but increases linearly as one moves up along the chain, reaching its maximum when \( s = L \) at the support point. Equation (10) is simply a restatement of the fact that for a triangular configuration the normal component of the gravitational force per unit length on the chain is exactly balanced by the surface tension.

5. Results and discussion

To explore the three configurations experimentally, we chose a silver chain of length 44.0 cm and mass 3.90 g. Our soap solution was a homemade mixture consisting of one part liquid dish detergent (Joy), two parts glycerin and seven parts distilled water. The chain was supported by a glass rod of diameter 0.5 cm.

It is worth noting that since the surface tension of typical soap solutions is in a range of 25–30 dynes cm\(^{-1}\), in order to meet the requirement of equation (10), the linear mass density of the chain, \( \lambda \), must be around 0.1 g cm\(^{-1}\) to admit a range of 50°–60° for the angle \( \theta \). So our choice of the chain was dictated by this requirement. Furthermore, we determined that the linear mass density of the chain does not change appreciably when it is coated with a soap film since any extra soap solution drains readily from the hanging chain.

To form each of the three configurations, first we wrapped the two ends of the chain around the glass rod with the midsections of the chain hanging down close together in a double strand (see figure 4). An eye dropper was used to drip several drops of the soap solution over the chain to cover the space between the rod and the hanging chain with a soap film. The excess soap solution was allowed to drain, after which, the two strands of the chain were slowly separated along the rod by increasing the distance between the two support points (designated as \( 2X \) in figure 1). When this distance is relatively small, the resulting configuration is the concave shape shown in figure 4.

When the distance \( 2X \) is increased further, the perfect triangular configuration is reached if the length of the hanging chain is not too long as shown in figure 5. A further increase of the distance \( 2X \) produces the convex shape shown in figure 6.
To reach these configurations, some practice is necessary to choose a suitable initial length for the hanging portion of the chain to achieve the desired configuration. In general, if the hanging part is too long, only the concave configuration of figure 4 is reached, with the lower part of the chain clinging together in a vertical line. So some adjustment of the length of the hanging chain is needed to reach the desired configuration. This is why the relative size of the hanging chain is different in the several pictures shown.

Returning to the triangular configuration of figure 5, we can readily measure the angle $\theta$ which is the only experimental parameter needed to obtain the surface tension of the soap solution. For accuracy, it is important to make sure that the camera is perpendicular to the plane of the hanging chain so that the angle of the chain in the picture reflects its real value. Of course the angle may also be determined directly by measuring the linear dimensions of the triangular configuration, but a picture is more convenient.
Figure 7. Picture of the chain in a mixed configuration. Most of the chain is wrapped around the supporting rod with the two ends free in the middle. A soap film covers the area between the supporting rod and the chain. The piece of chain on the left is longer and hangs vertically down below the contact point with the right piece. The right side is in the triangular configuration while the left side is in the concave configuration. Clearly there is no tension at the tip of the chain on the right as it just touches the left side with no change in its direction.

In figure 5, we measured the angle to be $53^\circ$ which can immediately be used in equation (5) to give

$$\sigma = (\lambda g \cos \theta)/2 = (3.9/44)(982)(\cos 53^\circ)/2 = 26.2 \text{ dynes cm}^{-1}. \tag{11}$$

This result is very close to the value of the surface tension for this soap solution measured by more conventional methods.

Finally, it is interesting to note that according to equation (4), in a triangular configuration, the tension in the chain goes to zero at the lower tip of the triangle. To explore this conclusion, we proceeded to form a mixed triangular–concave configuration by the following procedure.

The chain was wrapped around the glass rod in such a way that its two free ends were hanging down from the midpoint and touching one another (see figure 7). The hanging chain on the left was longer than the one on the right. An eye dropper was used to drip a few drops of the soap solution over the hanging chains to form a film between them. Now the support point of the short chain was pulled along the rod until the chain formed into a straight line. On the other side, the longer chain formed into a concave curve with its end section hanging straight down below the point of contact with the end tip of the short side. Figure 7 shows a picture of this mixed configuration. Clearly there is no tension at the tip of the straight chain on the right as it just touches the left side with no change in its direction.

Furthermore, in figure 7, the vertically hanging end of the long chain provides a ready reference for an accurate measurement of the angle $\theta$ which the right chain makes with the horizontal. Indeed the easily measured angle between the right chain and the vertical is the complement of the angle between the right chain and the horizontal. We measured $\theta$ from this
picture to be 53° which is the same value as used in equation (11) to get the surface tension of the soap solution.

In summary, we have shown that, for a chain hanging from a horizontal rod, three new and distinct configurations are possible if a soap film covers the area bounded by the chain and the rod. We have reviewed the conditions under which the chain assumes a concave, triangular or convex configuration. The deciding factor turns out, not surprisingly, to be the relative strength of surface tension to the gravitational force per unit length of the chain. The conditions under which the chain assumes a perfect triangular shape are discussed in greater detail and analysed to obtain the tension in the chain. The triangular configuration is especially intriguing to undergraduates and can be used as a simple experiment to measure the surface tension of the soap solution.

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