Position Vectors in 3-d: Student Difficulty With Spherical Unit Vectors in Intermediate E&M

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Introduction

- Student difficulties in introductory mechanics have been studied extensively (e.g. see any paper by McDermott, Schaffer, et. al. of the last 15 years) [1]
- Even student difficulties in introductory E&M have begun to be studied more extensively
- But there are few studies of student difficulties in intermediate E&M
- This poster presents results from a study of student difficulties with position vectors and spherical unit vectors in Griffith's level E&M [2]]

Motivation

- My students
  - –were very competent with spherical *coordinates* (r,θ,ϕ)
  - –but still didn't seem to "get" spherical *unit vectors* $\hat{r},\hat{\theta},\hat{\phi}$
- Goals of intermediate E&M:
  - –understand the math and physics of Maxwell's Integral equations (Electric & Magnetic)
  - –functionally understand (i.e. be able to set up) 3-d vector integrations to calculate fields & potentials:

$$\bar{E}(\bar{r}) = \tfrac{1}{4\pi\varepsilon_o}\iint_S \frac{\sigma(\bar{r}')da'}{\left|\bar{r}-\bar{r}'\right|^2}\cdot\frac{\bar{r}-\bar{r}'}{\left|\bar{r}-\bar{r}'\right|}$$

$$V(\bar{r}) = \tfrac{1}{4\pi\varepsilon_o}\int_C \frac{\lambda(\bar{r}')dl'}{\left|\bar{r}-\bar{r}'\right|^2}$$

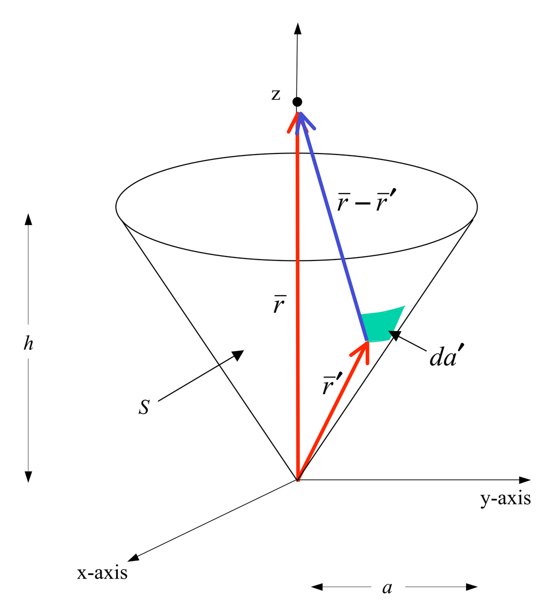
#1) Integrate over *source* charge distribution (λ, σ, ρ)
- modeled as point:
  - particles (λ)
  - patches (σ)
  - chunks (ρ)
- pointed to by $\bar{r}'$

#2) Find field or potential at the *field point* pointed to by $\bar{r}$

- Position vectors $\bar{r}$ and $\bar{r}'$ are **ubiquitous** in Maxwell's Integral equations
  - –students should be able to write down $\bar{r}$ and $\bar{r}'$ for any given non-Cartesian geometry
  - –start with spherically symmetric geometries

Example

A hollow cone of radius *a*, and height *h*, is centered on the *z*-axis with its tip at the origin and it's base in the $+z$ direction. It has uniform surface charge density, *σ*, on its curved sides, but no charge on its base. Find the electric potential at a point on the *z*-axis above the cone.



da' = *generic patch of source charge*

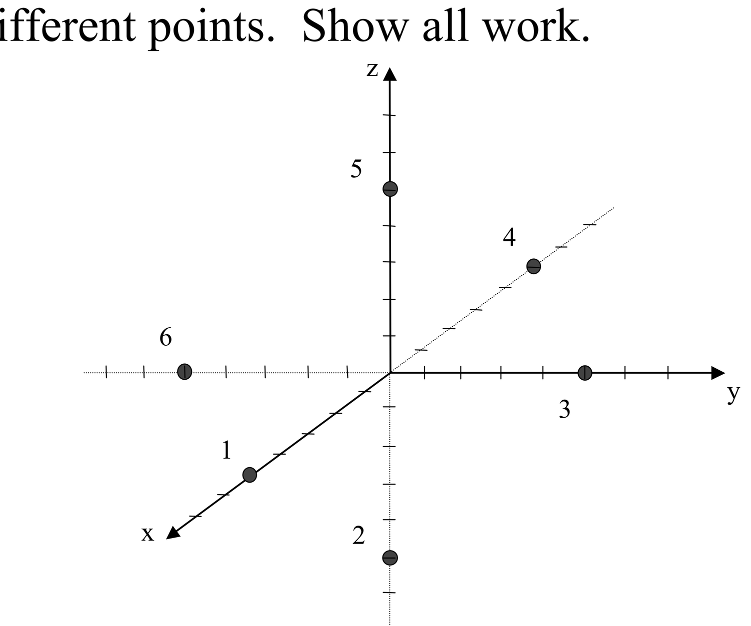
$\boxed{\bar{r} = \text{field point} = z\,\hat{z}}$ $\boxed{\bar{r}' = \text{points to patch} = r'\,\hat{r}'}$

$$\hat{r}' = \sin\theta'\cos\phi'\,\hat{x} + \sin\theta'\sin\phi'\,\hat{y} + \cos\theta'\,\hat{z}$$

$$\left\{\begin{array}{l} 0 < r' < \sqrt{a^2+h^2} \\ 0 < \varphi' < 2\pi \\ \quad\theta' = \tan^{-1}\left(a/h\right) \end{array}\right\}$$

Concept Test

Please write $\bar{r}$ in terms of $\hat{r},\hat{\theta}$, and $\hat{\phi}$ for the following six different points. Show all work.

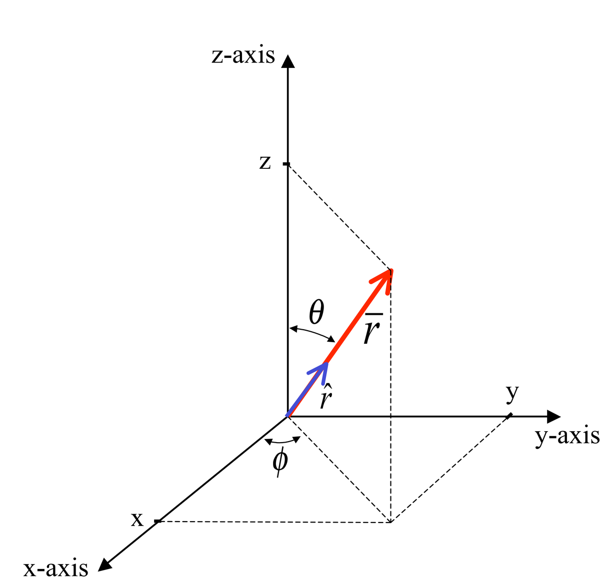


Expected Answer:

$\bar{r}_1 = 5\hat{r}$ $\bar{r}_2 = 5\hat{r}$ $\bar{r}_3 = 5\hat{r}$

$\bar{r}_4 = 5\hat{r}$ $\bar{r}_5 = 5\hat{r}$ $\bar{r}_6 = 5\hat{r}$

Explanation



$\boxed{\bar{r} = r\,\hat{r}}$

$$r = \sqrt{x^2+y^2+z^2}$$

$$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

$\hat{r}$ always points radially outward from the origin
- changes direction depending on θ and φ
- doesn't point in a constant direction like $\hat{x},\hat{y},\hat{z}$

For example, for point 1, θ=π/2 and φ=0, so

$$\hat{r} = \sin\tfrac{\pi}{2}\cos 0\,\hat{x} + \sin\tfrac{\pi}{2}\sin 0\,\hat{y} + \cos\tfrac{\pi}{2}\,\hat{z} = \hat{x}$$

While for point 2, θ=π (φ doesn't matter), so

$$\hat{r} = \sin\pi\cos\phi\,\hat{x} + \sin\pi\sin\phi\,\hat{y} + \cos\pi\,\hat{z} = -\hat{z}$$

Data Collection

TABLE 1. Schools from which data was collected and details of how the concept test was given at each school.

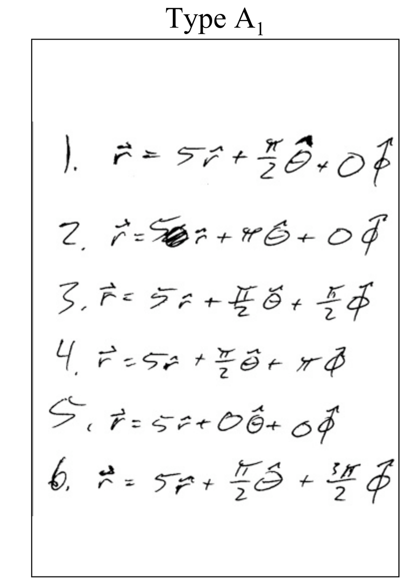
Institution	Textbook	N[a]	How Given	When Given
Small private liberal arts college in the upper midwest, **PLA**	Pollack & Stump [3]	12 of 12	As homework for credit	After completing both Chp2 (Vector Calculus) in class and relevant homework from Chp2
Small public university in the upper midwest, **SP**	Griffiths	6 of 6	Quiz	After completing both lecture on Section 1.4 (curvilinear coordinates) and relevant homework
Large public university in the southwest, undergraduates **LP-ug**	Griffiths	14 of 26	Volunteers who stayed after class	During the last week of a full year of intermediate E&M
Large public university in the southwest, graduate students **LP-g**	n/a	14 of 21	Volunteers who stayed after class	During the last week of the first year of graduate school, in their quantum course

[a] Indicates how many of the students officially registered for the course actually took the concept test.

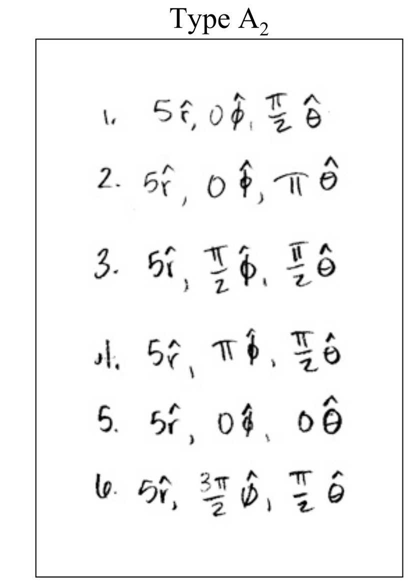
Typical Incorrect Answers

Explicitly include $\hat{r},\hat{\theta},\hat{\phi}$

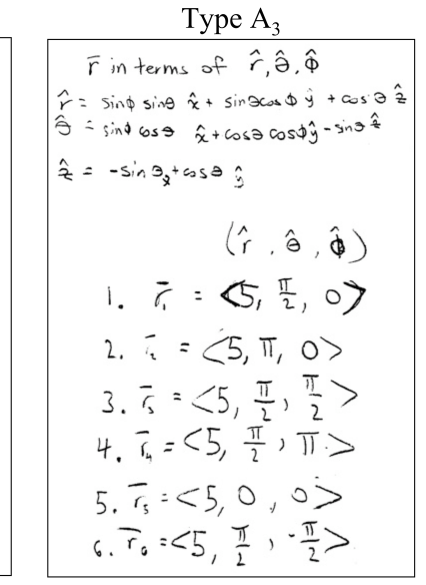
Type A_1



Type A_2

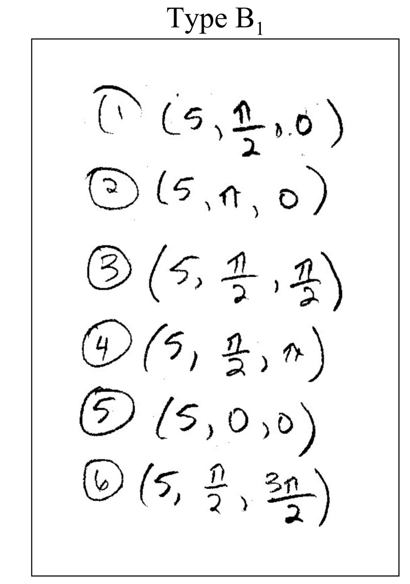


Type A_3

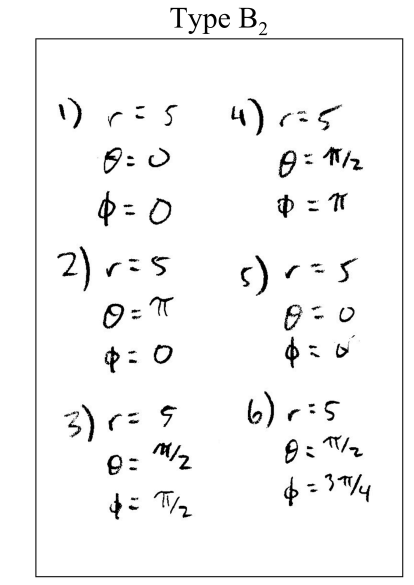


Do not include $\hat{r},\hat{\theta},\hat{\phi}$

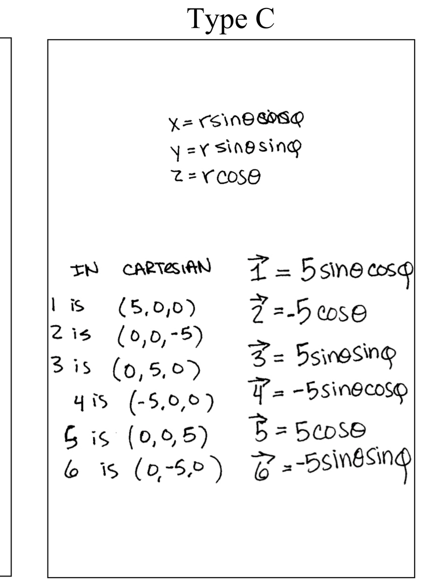
Type B_1



Type B_2



Type C



Results

TABLE 2. Results for each school and composite totals for the concept test shown previously

		Answer[b]								r, θ, ϕ all
School	N	Correct	A1	A2	A3	B1	B2	C	Other	correct[c]
PLA	12	0	6	2	1	2	1	0	0	6
SP	6	0	3	0	1	1	0	1	0	3
LP-ug	14	1	6	3	0	1	2	0	1	6
LP-g	14	0	6	1	1	4	0	2	0	7
	46 100%	1 2%	21 46%	6 13%	3 7%	8 17%	3 7%	3 7%	1 2%	22 48%

[b] Number of students at each school who made this kind of error.
[c] Number of students at each school who got all *r*, *θ*, and *ϕ* values correct for all six points.

Observations

- Only one person got the correct answer!
- Graduate students did no better than undergraduates
- The results don't seem to depend on text, teacher, year in school, type of school, class size, etc.
- Nearly half of all students put answer A_1
- About twenty percent just listed *r, θ, ϕ*
- Seventy percent (A+correct+other) explicitly included unit vectors in their answers
- Half of all student had difficulty determining correct *values* for the spherical coordinates themselves
- Students don't understand what it means to express a vector as a linear superposition of unit vectors (A_2, A_3)
- Students misapply methods from Cartesian coordinates (A_1), (i.e. pattern matching)

$$\bar{r} = (x,y,z) = x\,\hat{x} + y\,\hat{y} + z\,\hat{z} \qquad \bar{r} = (r,\theta,\phi) \neq r\,\hat{r} + \theta\,\hat{\theta} + \phi\,\hat{\phi}$$

Acknowledgements

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References

[1] L. C. McDermott and E. F. Redish, "Resource Letter: PER-1: Physics Education Research," *Am. J. Phys.* **67**, 755-767 (1999).

[2] D. J. Griffiths, *Introduction to Electrodynamics*, 3rd Edition, Upper Saddle River, New Jersey: Prentice Hall, 1999.

[3] G. L. Pollack and D. R. Stump, *Electromagnetism*, 1st Edition, San Francisco, California: Addison-Wesley, 2002.