

Student Difficulties in Understanding Probability in Quantum Mechanics

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Abstract. We have investigated student difficulties in understanding and interpreting probability and its relevant technical terms as it relates to quantum measurement. These terms include expectation value, probability density, and uncertainty. From this research, it is evident that students have difficulties in understanding these terms and often fail to differentiate among similar but different concepts. In addition, students' difficulties with the concepts of probability often interfere with their understanding and application of the Uncertainty Principle.

Keywords: physics education research, student difficulties, quantum mechanics, probability

INTRODUCTION

Since the beginning of history, in a world of uncertainty, man has pursued certainty. The mission of the philosophies of Plato and Aristotle was to discover true and certain knowledge. Later, with the development of Newtonian mechanics the deterministic worldview remained entrenched. It was only with the advance of thermodynamics that Boltzmann introduced the idea of probability into physical law. In the same way for most students, a quantum mechanics course is the first time a physical concept has been explained in terms of probability. It is often difficult for them to replace the basic convictions of a deterministic worldview with the probabilistic view of quantum mechanics.

In line with physics education research in Newtonian mechanics [1], there has been some work on student difficulties in learning the concept of probability in quantum mechanics. For example, Bao and Redish suggest that probabilistic interpretations of physical systems are a prerequisite to learning quantum physics [2]. Other studies investigated students' difficulties with probability in the context of quantum measurement. For instance, Styer [3] reported fifteen common misconceptions in quantum mechanics regarding quantal states, quantum measurement, and identical particles. Singh [4] has constructed a list of student misconceptions in the domain of quantum measurement and time evolution. In addition, Domert, et al. presented students' beliefs about probability in the context of quantum scattering and tunneling [5].

The understanding of probability is not limited to the topics of quantum mechanics. For example, many studies in mathematics education and cognitive psychology report student difficulties in this topic [6].

Recent developments in nanotechnology, photonics, and superconductivity bring to our everyday life advanced engineering and business devices that can be explained only through principles of quantum mechanics. The framework of quantum mechanics introduces the ideas of probabilities, wave-particle duality, and non-locality into the foundation of physics. These new ideas are abstract and usually difficult for introductory college students. Nevertheless, the overall research focused on student understanding of quantum probability is very limited.

MOTIVATION

During informal observations of quantum mechanics classes, in the study of students' written homework, and in one-on-one interactions, we observed that students often have trouble with the concepts of probability, probability density, and expectation value. For example, in many cases students described the "expectation value" and "probability density" in very vague terms and even used these two terms interchangeably. In addition, they had difficulties in constructing a physical meaning for quantitative values of these terms. For example, in a homework question [7], students needed to show that:

For a localized wavefunction at $x=x_0$ (a delta function) all values of momentum are equally probable.

A correct solution requires two steps. First, the momentum space representation of $\delta(x-x_0)$ results in $\varphi(k)=(1/2\pi)^{1/2}e^{-ikx_0}$, and the probability density of the momentum will be a constant value: $|\varphi(k)|^2=1/2\pi$. Second, one should conclude that a constant value for the probability density of momentum gives an equal probability for finding any value of momentum.

Students had difficulties with both parts of the solution. Many students were not able to calculate the probability density of the momentum correctly. In addition, their interpretation of their incorrect calculations revealed conceptual difficulties. Examples of the students' incorrect reasoning after calculating incorrect values for the probability density of momentum are discussed below:

S1: "Since $|\varphi(k)|^2 dk$ is the probability of finding the measurement between $\hbar k$ and $\hbar(k+dk)$, and here, $\varphi(k)=\delta(k)$ with $x=x_0$, this gives us $\langle p \rangle = \text{constant}$."

This student has difficulty distinguishing that the delta function in this problem represents the wavefunction in position space and not in momentum space. More importantly, the student went on to say that a "constant value for the expectation value of momentum" was the reason for having "equal probability" of measuring any value of momentum.

S2: "The expectation value of x is the Fourier transform of your delta function, which produces a constant $\langle p \rangle$, so you have the same probability of any value."

First, this student confuses the Fourier transform of the wavefunction with the expectation value of x . Second, he also interprets a "constant value for the expectation value of momentum" as the reason for "equal probability" in measuring any value of momentum.

Some students incorrectly calculated that $|\varphi(k)|^2=\infty$, and concluded that:

S3: "...this means that any value of momentum is probable."

In addition to calculating the incorrect value for the probability density, this student interprets the "infinite value for the probability density" as the reason for "equal probability" in measuring any value of momentum.

Another student calculated $\langle p \rangle = \infty$ incorrectly and stated that:

S4: "...therefore, the measurement of momentum can have any value."

In this case the student presents an "infinite value for the expectation value of the momentum" to be the reason for "equal probability" in measuring any value of momentum.

These initial observations motivated us to systematically investigate students' understanding of probability and related concepts.

OVERVIEW

This study is part of a systematic investigation of student learning of quantum mechanics that aims to determine the most common mathematical and conceptual difficulties students encounter in upper-division undergraduate quantum mechanics courses [8]. In this paper, we focus only on one specific physical situation in which students should be able to make appropriate interpretation of probability density, expectation value and uncertainty. This specific situation, when the probability density in position space is essentially a delta function and in momentum space is a constant, is an idealized situation that may present unusual difficulties in and of itself. In addition, we study students' ability to interpret different quantitative values for these terms to their respective meaning in physics.

We used a variety of methods in order to enhance our understanding of the nature of student difficulties in quantum mechanics. In addition to formal methods, such as individual interviews, written questions, and multiple-choice questionnaires, we have also collected data from informal methods such as class observation and studying students' exam and homework assignments. Of 21 weekly questionnaires in this study, five pertained to topics of probability. These were a series of related multiple-choice questions on a particular topic in which students' incorrect ideas from interviews and homework were the alternative responses.

Three lecture format classes were involved in this study. (1) The first quarter of upper-level undergraduate quantum mechanics (P631-03) in the fall of 2003, with 35 students and "Introduction to Quantum Mechanics" by Griffiths as their text. (2) The first quarter of upper-level undergraduate quantum mechanics (P631-04) in the fall of 2004, with 60 students and "Introductory Quantum Mechanics," by

Q1. Emma is solving a quantum problem. She is asked to show that for a particular system, any value of the momentum is equally probable. She needs to show that: (pick the best that applies)

- a) The probability density of the momentum is infinity.
- b) The probability density of the momentum is zero.
- c) The probability density of the momentum is a nonzero constant.
- d) The expectation value of the momentum is infinity.
- e) The expectation value of the momentum is zero
- f) The expectation value of the momentum is the same everywhere.

Liboff as the text. (3) Sophomore Introductory quantum mechanics (P263-04) in the spring of 2004, with 61 students and “Six Ideas that Shaped Physics” Unit Q by Moore as the text. Questions on probability densities, expectation values, and uncertainties in measurement were given to students in all three classes [9]. Questions **Q1** and **Q2** are examples of such.

DATA

The correct answer for **Q1** is choice (c). Although this question followed instruction on related topics, approximately 70% of the students in P263-04 and P631-03 and more than 50% of the students in P631-04 answered **Q1** incorrectly [See Table 1]. In addition, in all three classes over half of the incorrect choices were choice (f). This suggests some students may have confused “expectation value” with “probability density.”

Question **Q2** is a multiple-choice, multiple response question concerning related terminologies in probability. Due to the higher-level content of this question [10], we gave this question only to the students enrolled in the P631 courses. Students were also asked to explain their reasoning.

Q1	P263-04	P631-03	P631-04
Correct	30% (n=11)	31% (n=11)	45% (n=25)
Incorrect	70% (n=26)	69% (n=24)	55% (n=29)
Choice (f)	43% (n=16)	34% (n=13)	33% (n=18)
Total	37	35	55

TABLE.1. The Summary of students’ responses to Q1

Q2. Suppose that at time $t = 0$, a position measurement is made on a particle and $x = x_0$ is found. Assume that measurement was precise enough and the wavefunction immediately following it is well approximated by a δ -function (or a very narrow Gaussian)

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x_0)} dk$$

Which statements are correct about this system? Choose all that apply. Explain your reasoning.

- a) $\phi(k) = \infty$
- b) $|\phi(k)|^2 = 1/2\pi$
- c) $\phi(k) = 0$
- d) $\langle p \rangle = 0$
- e) $|\phi(k)|^2 = \text{nonzero constant}$
- f) $\langle p \rangle = \infty$
- g) $|\phi(k)|^2 = 0$
- h) $\langle p \rangle = \text{nonzero constant}$

The correct answers to **Q2** are choices (b), (d), and (e); nevertheless, none of the students picked the analytical answer, choice (b), and none of the students who picked choice (d), gave correct reasoning for this choice. Of the students who gave at least one correct response, only students who picked choice (e) (~10%) had also a correct reasoning [See Table 2]. An example of incorrect reasoning given by a student who chose (d) as his answer follows:

“Since a position measurement is precise enough to have a function $f(x)$ then to measure $\langle p \rangle$ with any amount of accuracy is almost null according to the Uncertainty Principle.”

This student reasoning shows a tendency to treat $\langle p \rangle$ as a measure of the “accuracy” or “certainty” in a momentum measurement rather than an average value that befits an expectation value. This reasoning fails to correctly apply the Uncertainty Principle.

Q2	P631-03	P631-04
At Least One Correct Choice	27% (n=6)	37% (n=20)
With Correct Reasoning	9% (n=2)	11% (n=6)
Incorrect	73% (n=16)	64% (n=34)
Total	22	54

TABLE 2. The Summary of students' responses to Q2

Many students picked choice (a) and (f) simultaneously. The first inference that could be made from students choosing these two choices together is that students view $\langle p \rangle$ and $\phi(k)$ the same. The second implication, considering some of the students' reasoning, is that these students fail to apply the Uncertainty Principle correctly. For instance, consider the following reasoning given by a student who chose (a) and (f) as his answers to Q2:

“a and f, because if you know the exact position, nothing can be known about the momentum because of the Uncertainty Principle.”

Here, this student interprets the “infinite value for expectation value” of momentum as “maximum uncertainty” in momentum measurement.

About one third of the students in both classes picked choice (h): $\langle p \rangle = \text{nonzero constant}$, with reasoning similar to the following:

“...because any value is possible.”

As discussed above, the expectation value of the momentum in this case is zero. It is possible that this student confused the “expectation value” with the “probability density” in momentum measurement that is indeed a nonzero constant in this problem ($1/2\pi$). In addition, some of the other students' reasons for choosing this answer show that they interpret a “constant expectation value” of the momentum as the “equal probability” for any momentum measurement. This problem is similar to our findings from Q1, where students confuse “expectation value” with “probability density” in measurement.

SUMMARY

Our study indicates that a large fraction of students have difficulties in understanding probability concepts and related terminologies. Students lack the ability to distinguish among similar but different concepts. For example, some students tend to confuse terms such as “expectation value”, “probability density”, and “uncertainty” in measurement and often interpret a “constant value for expectation value of momentum” as “equal probability in measuring different momenta.” There is evidence reported in this paper showing some students have serious difficulties in interpreting the quantitative outcomes for these terms into physical meaning. For example, some students tend to interpret $\langle p \rangle = 0$ or $\langle p \rangle = \infty$ as maximum uncertainty in momentum measurement. In addition, when students apply the Uncertainty Principle to a system they often confuse $\langle p \rangle$ with Δp .

The results of this study have implications not only for instructors but also for reform efforts towards improving the physics curriculum. First, there is a need for explicit instruction to address the conceptual and mathematical difficulties that students have with expectation values, probability densities and the Uncertainty Principle. These concepts are abstract and new for most students starting quantum courses. Explicit and clear definitions, applications and instructions are needed to help students understand and distinguish these terms. Research has shown that, unless there is explicit instruction, difficulties with fundamental concepts are likely to persist and prevent advanced learning [11]. Second, identifying student difficulties, in turn can guide the development of research-based instructional materials for addressing these difficulties. However, further investigation of these difficulties is necessary for finding effective ways to addressing them.

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