Improving Student Understanding of Quantum Mechanics

Chandralekha Singh, University of Pittsburgh
Mario Belloni and Wolfgang Christian, Davidson College

Richard Feynman famously stated that "nobody understands quantum mechanics"[1]. Of course, he was referring to the many strange, unintuitive foundational aspects of quantum theory, such as the origins of indeterminism or state reduction during measurement according to the Copenhagen interpretation. Despite these underlying fundamental mysteries, the theory has remained the cornerstone of modern physics. Most physicists have taken quantum mechanics as undergraduates, only to repeat it in their first year of graduate school. It is generally assumed that after all this instruction, students have become certified quantum mechanics, able to solve the Schrödinger equation, manipulate Dirac bras and kets, calculate expectation values, and most importantly, interpret their results in terms of real or thought experiments. The title of this article refers to this sort of functional understanding quite distinct from the foundational issues alluded to by Feynman.

Extensive testing and interviews demonstrate that a significant fraction of advanced undergraduate and beginning graduate students, even after one or two full years of instruction, still are *not* proficient at these skills. They often possess deep-rooted misconceptions about quantum formalism such as the meaning and significance of stationary states, meaning of expectation value, properties of wave functions, and quantum dynamics. Even students who excel at solving technically difficult questions are often unable to answer qualitative versions of the same questions.

A growing number of researchers have begun to investigate and address issues related to student understanding of quantum mechanics [2, 3, 4, 5], borrowing heavily from tools and methods developed for introductory level physics [6]. In this article we motivate the need for physics education research in quantum mechanics by illustrating some of the most pervasive difficulties students have. We then survey tools that are being developed to help improve and deepen student understanding of this important subject.

1 Universal Nature of Student Misconceptions

Quantum mechanics is often one of the last courses an undergraduate physics major takes. Instructors generally assume that by the time students enter their classroom, they have learned all of the problem solving, reasoning and self-monitoring skills they will need. Investigations support just the opposite conclusion [5, 7]. For example, while many students in advanced quantum mechanics courses learn to solve the time-independent Schrödinger equation with complicated potentials and boundary conditions, many display deep-seeded confusions about quantum dynamics that are largely conceptual in nature [3, 4, 5, 8, 9]. Misconceptions in other areas are abundant as well. Investigations by the authors [5] and others [8] show that most of these difficulties are *universal,* i.e., independent of background, teaching style, textbook, and institution. The patterns of incorrect notions are analogous to those that have been well-documented for introductory physics courses [6].

2 Examples of Student Misconceptions

Several studies have investigated misconceptions related to quantum mechanics in modern physics courses [3, 4, 9]. Here we illustrate examples of student difficulties in quantum mechanics, identified using a research-based written survey administered to eighty nine undergraduates and more than two hundred first-year physics graduate students from seven different universities [5], and followed up by numerous in-depth interviews. Topics covered by the survey include general properties of wave function, its time dependence, probabilities for measurement outcomes, expectation values for energy and other observables and their time dependence.

One question on the survey asked students to write down the most fundamental equation in quantum mechanics. Only 32% of students indicated the time-dependent Schrödinger equation $\hat{H}\Psi = i\hbar\, \partial\Psi/\partial t$ (or equivalently, defining commutation relations); by contrast, 48% of students wrote down the time-independent Schrödinger equation: $H\Psi = E\Psi$. We note that Schrödinger received his Nobel prize for the former, not the latter, equation [10]. As we shall see in the questions below, students' obsession with $\hat{H}\Psi = E\Psi$ is closely linked to a number of difficulties and misconceptions about quantum mechanics.

Question 1: Consider the following statement: "By definition, the Hamiltonian acting on *any* allowed state of the system Ψ will give the same state back, i.e., $\hat{H}\Psi = E\Psi$, where E is the energy of the system." Explain why you agree or disagree with this statement.

Figure 1 summarizes the responses that students gave, along with their "branching ratios". The correct answer was given by 29% of students: the statement is true only if Ψ is a stationary state. Among the incorrect answers, a full 39% wrote that the statement is true unconditionally. Typically, these students were supremely confident of their answers. For example, one student wrote: "Agree. This is what 80 years of experiment has proven. If future experiments prove this statement wrong, then I'll update my opinion on this subject."

A total of 11% of the students believed that any statement involving a Hamiltonian H acting on a state Ψ constitutes a measurement of its energy. Armed with this philosophy, some agreed with the statement, believing that $H\Psi$ should yield $E\Psi$, rendering the statement true. Others disagreed, stating that the right side of the equation has collapsed after the measurement so $H\Psi = E_n\psi_n$ where ψ_n is the stationary state to which the wavefunction collapsed.

A fraction of students (10%) claimed that the statement is true only if H is not explicitly time-dependent, and that for time-independent Hamiltonians the energy of the system is always conserved and hence $H\Psi = E\Psi$. This statement is incorrect since Ψ need not be an eigenstate of H, even if H is time-independent.

This next question illustrates student misconceptions about quantum dynamics in an infinite square-well potential and interpretation of expectation value.

Question 2: The wave function of an electron in a one-dimensional infinite square well of width a, $x\epsilon(0,a)$, at time $t = 0$ is given by $\Psi(x,0) = \sqrt{2/7}\phi_1(x) + \sqrt{5/7}\phi_2(x)$, where $\phi_1(x)$ and $\phi_2(x)$ are the ground state and first excited stationary state of the system. ($\phi_n(x) = \sqrt{2/a}$ $sin(n\pi x/a)$, $E_n = n^2\pi^2\hbar^2/(2ma^2)$ where $n = 1, 2, 3...$) (Note: A potential energy diagram was provided.)

(a) Write down the wave function $\Psi(x,t)$ at time t in terms of $\phi_1(x)$ and $\phi_2(x)$.

(b) You measure the energy of an electron at time $t = 0$. Write down the possible values of the energy and the probability of measuring each.

(c) Calculate the expectation value of the energy in the state $\Psi(x,t)$ above.

Student answers to these questions are summarized in **Figure 2**. In response to Question 2(a), only 43% of the students provided a correct response. (Note that responses with incorrect intermediate steps were considered correct). The most common mistake (31%) was to write a common phase factor for both terms, *e.g.*, $\Psi(x,t) = \Psi(x,0)e^{-iEt/\hbar}$. (There were many other varieties of common phase factors represented as well.) Follow-up interviews confirm that students had trouble distinguishing the properties of stationary and non-stationary states. Several students thought that time dependence took the form of a decaying exponential. During interviews, some of them explained their choice by insisting that the wave function must decay with time because "that is what happens for all physical systems". Interestingly, 9% of the students wrote that $\Psi(x,t)$ should not have *any* time dependence whatsoever; during interviews, some tried to justify their claim by pointing out that the Hamiltonian is time-independent.

Although Question 2(b) was the easiest on the survey with 67% correct responses, comparison with the response for Question 2(c) is particularly revealing. It shows that students can calculate probabilities for the outcome of measurements, but that many were unable to use that information to determine an expectation value. Many students who answered 2(b) correctly (including those who also answered (c) correctly) calculated $\langle E \rangle$ by brute-force methods: first writing $\langle E \rangle = \int_{-\infty}^{+\infty} \Psi^* \hat{H} \Psi dx$, then expressing $\Psi(x,t)$ in terms of the two energy eigenstates, then acting $\hat{H}$ on the eigenstates, and finally using orthogonality to obtain the final answer. Some got lost early in this process while others did not remember some other mechanical step, e.g., taking the complex conjugate of the wave function, using orthogonality of stationary states or not realizing the proper limits of the integral.

Interviews reveal that many did not know or recall the interpretation of expectation value as an ensemble average, and did not realize that expectation values could be calculated more simply in this case by taking advantage of their answer to Question 2(b). Some believed that the expectation value of energy should depend on time (even those who correctly evaluated $\Psi(x,t)$ in Question 2(a)).

In another question, students were given a potential energy diagram for a finite square well and asked to draw qualitative sketches of (I) the ground state wavefunction and (II) any one scattering state wavefunction. They were asked to comment on the shape of the wave function in all the three regions. Only 57% provided the correct response for part (I) and 17% for part (II). Even students who had performed well on quantitative questions on related topics in homework and examinations and imposed the boundary conditions correctly, sketched wavefunctions with discontinuities or cusps. Roughly 10% of students incorrectly sketched exponentially decreasing wavefunction inside the well. In part (II), several students made comments such as "the particle is bound inside the well but free outside the well". The comments displayed confusion about what "bound state" means and whether the entire wavefunction is associated with the particle at a given time or the parts of the wavefunction outside and inside the well are associated with particle at different times.

Other questions posed to students confirmed that many of the difficulties students have are both conceptual and deep-rooted. For example, analogous difficulties were also observed in response to conceptual questions about Larmor spin precession, especially with regard to the expectation values of spin components and their time dependence, given a particular initial state [5]. Interestingly, course instructors were surprised and noted that on similar but exclusively quantitative calculations, student performance was significantly higher. In this sense, qualitative understanding of quantum mechanics is much more challenging than facility with the technical aspects. Similar to "plug and chug" problems in introductory physics, strict quantitative exercises in quantum mechanics often fail to provide adequate opportunity for reflection upon the problem solving process and drawing meaningful inferences. As discussed later below, learning tools that combine quantitative and qualitative problem solving can be effective in helping students learn quantum mechanics.

3 Origins of Student Misconceptions

Why is student *misunderstanding* of quantum mechanics concepts "quantized", i.e., collapsed into a few distinct patterns, independent of institution, textbook or instructor? The universal nature of student misconceptions about quantum mechanics is somewhat surprising, given the abstract nature of the subject. Misconceptions in introductory-level physics are often viewed as originating from incorrect world-views (e.g., watching too many Road Runner cartoons). But it is hard to explain conceptual difficulties in quantum mechanics in this fashion.

Shared misconceptions in quantum mechanics can be traced in large part to incorrect over-generalizations of concepts learned earlier [5]. The following examples show difficulties which are due to correct or incorrect over-generalizations from classical physics:

- Many students believe that if the expectation value of a physical observable is zero in the initial state, its expectation value cannot have any time-dependence. This misconception is an over-generalization of a similar misconception in introductory physics that if the velocity of an object is zero, its acceleration must be zero as well. (Of course, if this were true, a car could never travel from rest.)
- Many students believe that an object with a label x is orthogonal to or cannot influence an object with a label y, e.g., they believe that eigenstates of $\hat{S}_x$ are orthogonal to eigenstates of $\hat{S}_y$. This overgeneralization originates from properties of classical vectors.

A second mechanism is failure to distinguish between closely related concepts. Below are two examples of commonly held (mistaken) beliefs about quantum mechanics that are consistent with many of the incorrect answers given by students.

- Many students believe that the expectation value of *any* Hermitian operator is time-independent if the initial state is an eigenstate of that operator. This belief can be regarded as the over-generalization of two true statements: (1) The expectation value of any Hermitian operator is time-independent if the state is a stationary state, and (2) The statement is true for any state if the operator commutes with the Hamiltonian.
- Many students believe that after measurement of *any* physical observable, the system gets "stuck" in that eigenstate forever unless an external perturbation is applied. Again, the statement is true only for observables whose operators commute with the Hamiltonian, but students seem to have overgeneralized this property of measurement to include all observables. (Incidentally, some students believe the opposite, i.e., if one waits long enough, the time-evolution will guarantee that the wavefunction will go back to the original state before the measurement.)

4 Improving Student Understanding

How can instruction in quantum mechanics be modified to reduce student difficulties? As we will see below, being precisely aware of the difficulties students typically face while learning quantum mechanics constitutes an important first step in developing strategies that can help students. In the section below, we will discuss learning tools available for quantum mechanics and how tutorials embedded within a coherent curriculum are well suited to direct targeting of the misconceptions that have been identified through research.

4.1 Quantum Visualization Tools

Fast and realistic computer-based visualization tools can play a key role as effective learning tools in a variety of advanced-level topics including quantum mechanics. The computational power of common desktop and laptop computers is more than sufficient to run these simulations, and many heretofore inaccessible topics (including those from current experimental and theoretical research) in quantum mechanics can now be discussed and exemplified in concrete terms. Most can be run within web browsers and are written using meta-languages and virtual machines that ensure that the software is platform-independent and adaptable.

One of the first quantum simulations for instructional purposes was created in 1967 by Goldberg [11], who simulated scattering of Gaussian wave packets off of different wells and barriers. Computer-generated probability densities were displayed on a cathode-ray tube, photographed, and then the successive frames were turned into a movie. The images and films that resulted have been referenced and/or reprinted in numerous textbooks, even to this day. The "picture" books of S. Brandt and H. Dahmen [12] and the *Visual Quantum Mechanics* books by Thaller [13] continue this approach, including more scenarios depicting time development with successive images or using *QuickTime* movies, respectively. This work was extended by the Consortium for Upper-level Physics Software (CUPS) Series quantum mechanics book by Hiller Johnston, and Styer [14], and more recently by the book *Physlet Quantum Physics* [15], both of which extensively use computer-based interactive simulations.

More recently, the Open Source Physics (OSP) project [16] has developed a freely available and open source Java library and easy-to-use vocabulary for the development of physics software and the distribution of Web-based curricular material. This technology, originally designed for the teaching of computational physics, enables the development of curricular materials by allowing the user to change and store initial conditions and associated narrative for simulations. These exercises can then be combined into a topical unit, and multiple topical units can be combined into a curriculum module. The OSP framework has been used to write and organize curricular material from computational physics, classical mechanics, electromagnetism, statistical physics, and quantum mechanics. (The OSP code library, documentation, and curricular materials can be downloaded from http://www.opensourcephysics.org.) The materials for quantum mechanics span the curriculum from introductory topics to more advanced research-based topics such as quantum-mechanical revivals and fractional revivals, shown in **Figure 3**, which have been of recent theoretical and experimental interest [17]. This example uses the OSP QMSuperposition program which inputs the expansion coefficients of a superposition of energy eigenstates (here a Gaussian wave packet in the infinite square well) and time evolves the state according to $\Psi(x,t) = \sum_n c_n \psi_n(x) e^{-iE_n t/\hbar}$. This analytic time evolution allows for a fast and accurate depiction of the state even at long time scales at which numerical solutions often fail.

In addition to "classic" simulations of quantum-mechanical time evolution, many computer-based simulations have been developed that cover more contemporary topics such as quantum information, as well as foundational issues that are usually skipped in standard sequences. For example, the OSP SPINS program extends David McIntyre's open source Java applet [18] by allowing simulated experiments to be stored and run easily. The user interface can be customized making the program useful for focused tutorials such as the simulation of single or multiple measurements on spin-1/2 particles or even a spin-1/2 interferometer used to simulate a quantum-mechanical Which-Way? (Welcher-Weg) experiment.

4.2 Quantum Tutorials

Teaching with technology, without a sound pedagogy, does not automatically yield significant educational gains. Evidence suggests that students can benefit from the same types of interventions that have proved successful at introductory levels [6]. In particular, activities that engage students and force them to challenge their beliefs and understanding of quantum mechanics have the greatest chance of success. Tutorials provide carefully developed activities that do not require a total revamping of instructional methods to have an impact; rather, they can be effective supplements to traditional instruction.

The tutorial approach consists of three main components:

- Student difficulties (determined from prior research) are first elicited by posing carefully designed tasks
- Students are guided through tasks designed to overcome these difficulties and organize their knowledge.
- Support is decreased gradually as students develop self-reliance.

The following features of tutorials make them particularly suited for teaching quantum mechanics:

- They are based upon research in physics education and pay particular attention to cognitive issues.
- They employ visualization tools to help students build physical intuition about quantum phenomena.
- They consistently keep students actively engaged in the learning process by asking them to predict what should happen in a particular situation and then providing appropriate feedback.
- They attempt to bridge the gap between the abstract quantitative formalism of quantum mechanics and the qualitative understanding necessary to explain and predict diverse physical phenomena
- They can be used in class by instructors once or twice a week as supplements to lectures as group activity or outside of the class as homework or as a self-study tool by students.
- They consist of self-sufficient modular units that can be used in any order that is convenient.

Daniel Styer [19] and Zollman et al. [3] have proposed that quantum concepts be introduced much earlier in physics course sequences than is traditional. To assist instructors, Zollman et al. [3] have developed the "Visual Quantum Mechanics" (VQM) suite, targeted to high school and college students, which integrates interactive visualizations with inexpensive materials with tutorial worksheets in an activity-based environment. Examples of instructional units in VQM include quantum tunneling, luminescence, laser adventure, and potential energy diagrams. **Figure 4** shows an example from VQM [20] that lets students explore scattering problems in one dimension. Students can calculate transmission coefficients for one-dimensional scattering potentials, view scattering states, and compute probability densities. The quantities are expressed in real units, yet students can change, for example, Planck's constant to observe its consequence for this problem.

Redish et al. [4] have developed "A new model course in applied quantum mechanics," a collection of resources incorporating tutorials and visualization tools to teach modern physics concepts to science and engineering students. Examples of instructional units [20] include the photoelectric effect, wave-particle duality, spectroscopy, shape of wave function, light-emitting diodes, band structure theory, and quantum models of conductivity. In the band structure tutorial [20], students are first asked to "solve" the time-independent Schrödinger equation for a single finite square well potential, using a program called "Energy Band Creator". A second well is then added, and students observe hybridization or bonding/anti-bonding states. Then, a larger number of "atoms" are considered, and students discover that the allowed energy states form bands (see **Figure 5**).

We have been developing and evaluating Quantum Interactive Learning Tutorials (QuILTs) [5], which are designed for upper-level undergraduate and beginning graduate students. They have been designed to engage students actively in the learning process and help them build links between the abstract formalism and the conceptual aspects of quantum physics, without compromising the technical content. QuILTs are based on the exemplary introductory physics tutorials developed by the University of Washington group [21] and combine conceptual and quantitative problem solving.

The topics for QuILTs cover fundamental as well as contemporary areas. Specific topics covered so far include the time-dependent and time-independent Schrödinger equation, expectation values, time-development of the wave function, measurement of observables, the uncertainty principle, the double-slit experiment and which-path information, Mach-Zehnder interferometers, Stern-Gerlach experiments, Larmor precession of spin, quantum key distribution, spin-1/2 systems, and product spaces.

Below we discuss excerpts of a QuILT designed to deal with time development [22]. The QuILT begins by asking students to predict the time dependence of two states, one stationary and the other non-stationary, for an electron in an infinite square well. If students choose an overall phase factor for the non-stationary state wave function, they are asked to predict the probability density at time t, i.e., the absolute square of the wave function. As described above, more than half of the students will choose an incorrect answer. The tutorial then asks students to compare their predictions against a computer simulation (**Figure 6**) for the time-evolution of the probability densities for those cases. The simulations have been adapted from the Open Source Physics simulations [16].

Watching the probability density not vary with time for the stationary state (**Figure 6a**) but explicitly vary with time for the non-stationary state (**Figure 6b**), students are forced to confront any inconsistencies in their thinking. Research on learning suggests that when students reach this juncture, they are challenged to resolve the discrepancy between their initial prediction and observation, and that students are highly receptive to guidance that will help them resolve their conceptual difficulties and build a more robust knowledge structure. Within an interactive environment, students then learn by example that the Hamiltonian governs the time evolution of the wavefunction so the eigenstates of the Hamiltonian are special with regard to time evolution, but that not all allowed states are energy eigenstates. Students must answer questions that relate to a variety of concrete examples to help organize their knowledge, and they receive prompt feedback. Later, concept maps are introduced that summarize the connection between different relevant concepts and principles. Then, students work on "paired" problems which use similar concepts as the tutorial problems but for which the contexts are different and no help is provided. The "paired" problems serve the dual purpose of helping students develop self-reliance and providing instructors feedback on the effectiveness of QuILTs.

The following is a brief excerpt from a quantitative QuILT on spin-1/2 systems. The tutorial breaks down the overarching problem into multiple-choice questions. In the full implementation, students will choose an answer on a computer, and are directed to appropriate feedback based on their choices.

Question 18: For a spin-1/2 system with the Hamiltonian $H_0 = C(\hat{S}^2 - \hat{S}_z^2)$ (where C is a constant), which one of the following is a basis that will yield $\hat{H}_0$ as a diagonal matrix?

Choices: (a) $|m_x, m_z\rangle$, simultaneous eigenstates of $\hat{S}_x$ and $\hat{S}_z$. (b) $|s, m_z\rangle$, simultaneous eigenstates of $\hat{S}^2$ and $\hat{S}_z$. (c) $|s, m_x\rangle$, simultaneous eigenstates of $\hat{S}^2$ and $\hat{S}_x$. (d) $|s, m_n\rangle$, simultaneous eigenstates of $\hat{S}^2$ and $\hat{S}_n$ where n points in an arbitrary direction in space.

Feedback: (a) No. It is impossible to construct such a state which is a simultaneous eigenstate of $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$ because these operators do not commute with each other. (b) Correct! In the basis consisting of the simultaneous eigenstates of $\hat{S}^2$ and $\hat{S}_z$, the Hamiltonian is a diagonal matrix. (c) No. The Hamiltonian does not commute with $\hat{S}_x$ whose eigenstates are characterized by quantum number m_x. (d) No. Although $\hat{S}^2$ and $\hat{S}_n$ commute with one another, in general, an eigenstate of $\hat{S}_n$ is not automatically an eigenstate of $\hat{S}_z$ (and hence $\hat{H}_0$).

4.3 Real Quantum Experiments

Over the last few years, many new experimental labs have been initiated that expose students to a variety of contemporary topics. For example, Galvez et al.[23] describe a set of five experiments in quantum optics that can be set up for about $35,000. Topics include single-photon self-interference, quantum erasure, and projective quantum measurements. Havel et al. [24] describe a number of quantum information processing experiments that can be performed using standard and widely available NMR spectrometers. Such experiences can serve to cement student knowledge and understanding of quantum mechanics as well as stimulate interest in relating quantum mechanics to "student observables". Greenstein and Zajonc's fascinating book [25], suitable for undergraduate courses as well as practicing scientists, is full of descriptions of experiments related to foundations of quantum mechanics and their possible interpretations.

5 Online Resources

The dissemination power of the World Wide Web through powerful search engines, databases, and more recently, digital libraries has made it easy to make effective educational tools widely available. In physics and astronomy, the stewardship role within the National Science Digital Library (NSDL, http://nsdl.org) is held by the ComPADRE digital library. ComPADRE stands for Communities for Physics and Astronomy Digital Resources in Education (`http://www.compadre.org`). This digital library is a result of the partnership of the AAPT, AAS, AIP/SPS, and APS and funding from the National Science Foundation. ComPADRE's goal is to aid teachers and learners in finding and using high-quality resources tailored to their specific needs. The ComPADRE digital library consists of several collections. The quantum mechanics collection is called The Quantum Exchange with the url: `http://thequantumexchange.org`. What makes ComPADRE particularly effective is that the material is peer reviewed, contains cross links with other ComPADRE collections such as the physics education research collection, and has a federated search feature (ability to perform one search in multiple libraries/databases/collections). This combined search engine allows a user to search for learning materials from multiple online collections and receive integrated results with a single search request. Collections currently searched are ComPADRE, MERLOT, the Physlet/OSP Database, the PADS Database, and the collections harvested by the NSDL. For example, all of the quantum mechanics materials developed by the Open Source Physics (OSP) project is accessible via ComPADRE Quantum Exchange search: `osp` or `open source physics` and has an easy to use vocabulary which is appropriate for digital libraries such as ComPADRE.

6 Summary

The teaching and learning of quantum mechanics currently stands at the fortuitous crossroads of advances in experimental, theoretical, computational and education research. Physics education research has matured beyond the introductory level, and researchers have begun to investigate and improve the quality of student understanding of quantum mechanics. The guidance provided by research-based learning tools has the potential to increase the number and proficiency of students, especially women and other underrepresented groups, to pursue advanced degrees and careers in physical science and engineering.

Figure Captions

Figure 1: Breakdown of student response to Question 1.

Figure 2: Breakdown of student response to Question 2.

Figure 3: An initially ($t = 0$) localized Gaussian wave packet ($x_0 = L/2$ and $p_0 = 80\pi$) in a one dimensional infinite square well is visualized at one-quarter the revival time: $T_{rev}/4$. By this time (for the simulation we have set $L = \hbar = 2m = T_{rev} = 1$), the wave packet has already spread encompassing the entire well only to reforms into two highly correlated copies of itself at the center of the well, but with each copy having the opposite momentum distribution. This "Schrödinger cat" state is characteristic of fractional revivals, while at the revival time the packet will exactly reform to its $t = 0$ configuration. The packet is visualized using both the position- and momentum-space wave functions (using magnitude-phase representation where the color represents the phase of the wave functions), the position-space probability density, a "quantum carpet" (which shows the history of the packet's evolution, past to present along the t axis and where brightness represents amplitude), and the expectation value of x and p with time.

Figure 4: An example from Visual Quantum Mechanics [3, 20] that lets students explore scattering in one dimension.

Figure 5: An example from Ref. [20] that lets students explore energy bands. Students can start with one energy well, then add more wells and observe changes in the energy levels. With the 10 wells shown in the figure, one can observe the emergence of energy bands (shown as horizontal strips).

Figure 6: An example from the QuILT [22, 16] that lets students learn that the probability density for the stationary state does not depend on time (see the time-lapse pictures in Figure 6(a)) but the probability density for the non-stationary state depends on time (see the time-lapse pictures in Figure 6(b)).

References

[1] R. Feynman, *The Character of Physical Law*, MIT press, Cambridge, MA 1965.

[2] See for example, the theme issue of Am. J. Phys. **70(3)**, 2002 published in conjunction with Gordon conference on teaching and research in quantum mechanics.

[3] D. Zollman, S. Rebello, K. Hogg, *Quantum Physics for everyone: hands-on activities integrated with technology*, Am. J. Phys., **70(3)**, 252 (2002); P. Jolly, D. Zollman, S. Rebello, A. Dimitrova, *Visualizing potential energy diagrams*, Am. J. Phys., 66(1), 57, 1998; *Research on Teaching and Learning of Quantum Mechanics, Papers Presented at the National Association for Research in Science Teaching*, http://perg.phys.ksu.edu/papers/narst/ (1999).

[4] L. Bao and E. Redish, *Understanding Probabilistic Interpretations of Physical Systems: A Prerequisite to Learning Quantum Physics*, Am. J. Phys., **70(3)**, 210-217 (2002); M. Wittmann, R. Steinberg, E. Redish, *Investigating student understanding of quantum physics: spontaneous models of conductivity* , Am. J. Phys. **70(3)**, 218-226, (2002).

[5] *Student understanding of quantum mechanics*, C. Singh, Am. J. Phys., **69 (8)**, 885-896, (2001); C. Singh, *Transfer of learning in quantum mechanics*, Proceedings of the Phys. Ed.

Res. Conference, Sacramento, CA, AIP, (P. Heron, S. Franklin, J. Marx Eds.), (2004); C. Singh, *Improving student understanding of quantum mechanics*, Proceedings of the Phys. Ed. Res. Conference, Salt Lake City, UT, AIP, (P. Heron, S. Franklin, J. Marx Eds.), (2005).

[6] L. C. McDermott, *Research on conceptual understanding in mechanics*, Phys. Today, **37**, 24, (1984); F. Reif, *Scientific approaches to science education*, Phys. Today **39**, 48, (1986); P. Laws, *Calculus-based physics without lectures*, Phys. Today, **44**, 12, (1991); E. Redish, R. Steinberg, *Teaching physics: figuring out what works*, Phys. Today, **52**, 20, (1999).

[7] E. Cataloglu and R. Robinett, *Testing the development of student conceptual and visualization understanding in quantum mechanics through the undergraduate career*, Am. J. Phys. **70**, 238-251 (2002).

[8] D. Styer, *Common misconceptions regarding quantum mechanics*, Am. J. Phys. **64**, 31-34, (1996).

[9] H. Fischler, M. Lichtfeldt, *Modern physics and students' conceptions*, Int. J. Sci. Ed., **14(2)**, 181-190, (1992).

[10] E. Schrödinger, *Quantisierung als Eigenwertproblem*, Annalen der Physik **384**, 361-76 (1926).

[11] A. Goldberg, H. M. Schey, and J. L. Schwartz, *Computer-generated Motion Pictures of One-dimensional Quantum-mechanical Transmission and Reflection Phenomena*, Am. J. Phys. **35**, 177-186 (1967).

[12] S. Brandt and H. Dahmen, *The Picture Book of Quantum Mechanics*, Springer-Verlag, New York, (2001).

[13] B. Thaller, *Visual Quantum Mechanics*, Springer-Verlag, New York, (2000).

[14] J. Hiller, I. Johnston, D. Styer, *Quantum Mechanics Simulations, Consortium for Undergraduate Physics Software*, John Wiley and Sons, New York, (1995).

[15] M. Belloni, W. Christian and A. Cox, *Physlet Quantum Physics*, Pearson Prentice Hall, Upper Saddle River, NJ, (2006).

[16] H. Gould, J. Tobochnik and W. Christian, *An Introduction to Computer Simulation Methods: Applications to Physical Systems*, 3/e, Addison Wesley 2007; W. Christian, *Open Source Physics: A Users Guide with Examples*, Addison Wesley 2007.

[17] H. Katsuki, H. Chiba, B. Girard, C. Meier, K. Ohmori, *Visualizing Picometric Quantum Ripples of Ultrafast Wave-Packet Interference*, Science, 311, 1589-1592 (2006).

[18] http://www.physics.orst.edu/%7Emcintyre/ph425/spins/spinsapplet.html The original Spins program was written by Daniel Schroeder and Thomas Moore for the Macintosh and was ported to Java by David McIntyre of Oregon State University and used as part of the Paradigms project. Both of these versions were, and remain, open source.

[19] D. Styer, *The strange world of quantum mechanics*, Cambridge Univ. Press, (2000).

[20] available at http://www.physics.umd.edu/perg/qm/qmcourse/NewModel

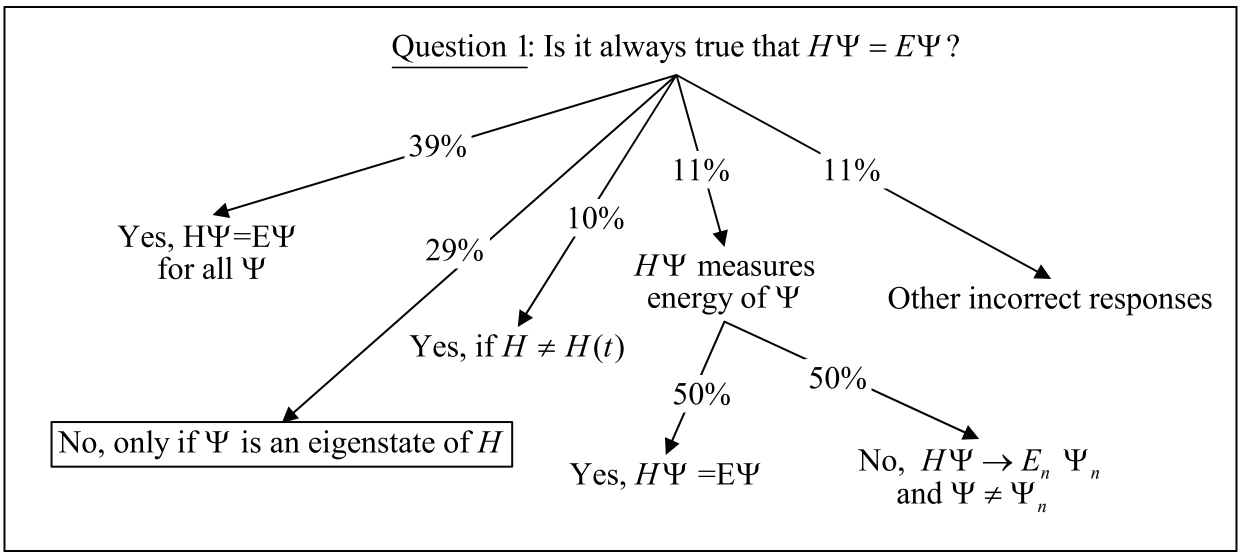
[21] L. McDermott, P. Shaffer, and the Physics Education Group, University of Washington, "Tutorials in introductory physics", Prentice Hall, NJ, (1998).

[22] Interested readers may view the full tutorial, available at http://www.opensourcephysics.org/publications/occam/quilt.html

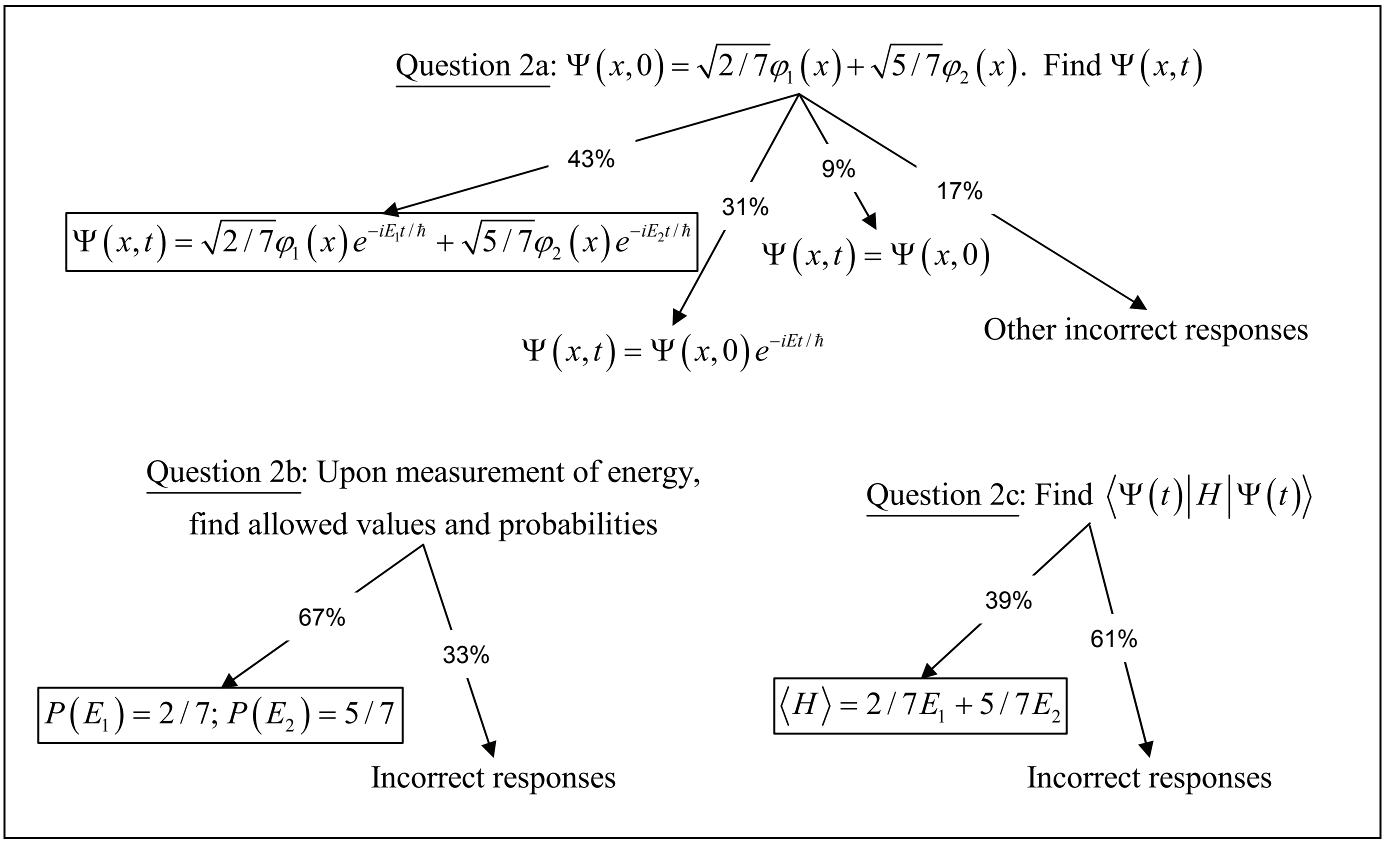
[23] E. J. Galvez, C. H. Holbrow, M. J. Pysher, J. W. Martin, N. Courtemanche, L. Heilig and J. Spencer, *Interference with correlated photons: Five quantum mechanics experiments for undergraduates*, Am. J. Phys. **73**, 127-140 (2005).

[24] T. F. Havel, D. G. Cory, S. Lloyd, N. Bouland, E. M. Fortunato, M. A. Pravia, G. Teklemariam, Y. S. Weinstein, A. Bhattacharyya, and J. Hou, *Quantum information processing by nuclear magnetic resonance spectroscopy*, Am. J. Phys. **70 (3)**, 345-62 (2002).

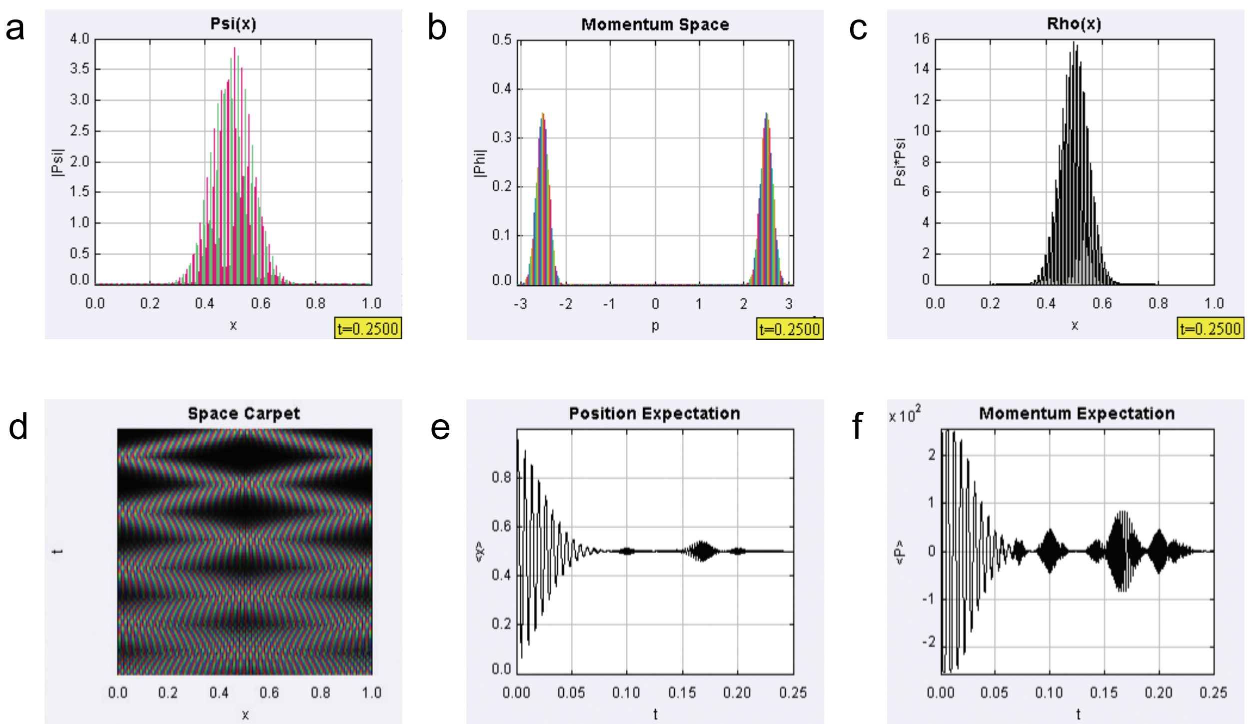
[25] G. Greenstein and A. Zajonc, *The quantum challenge: Modern research on the foundations of quantum mechanics*, Jones and Bartlett Publishers, Sudbury, MA, (1997).



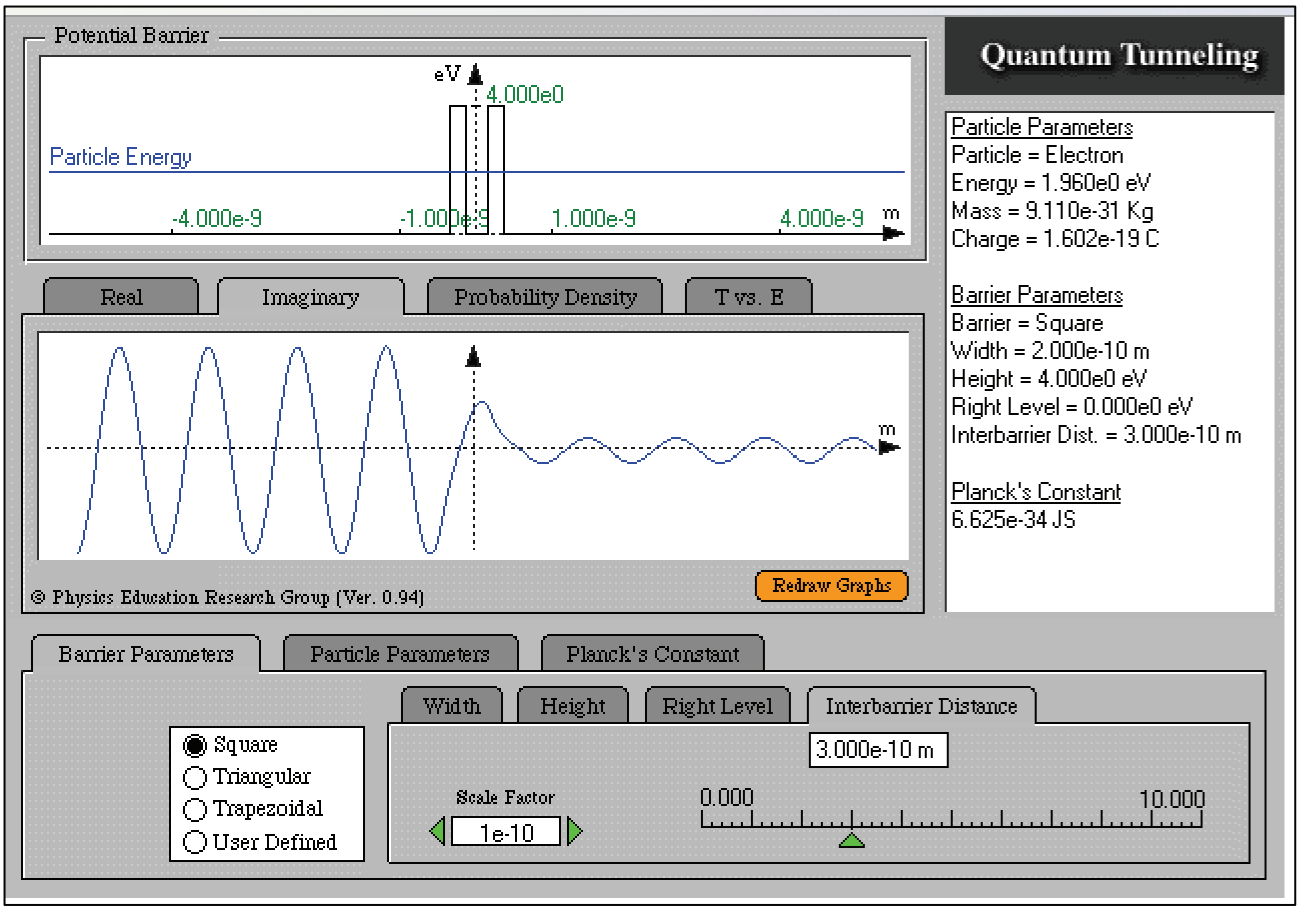
Singh, Belloni and Christian, "Improving Student Understanding of Quantum Mechanics", Figure 1



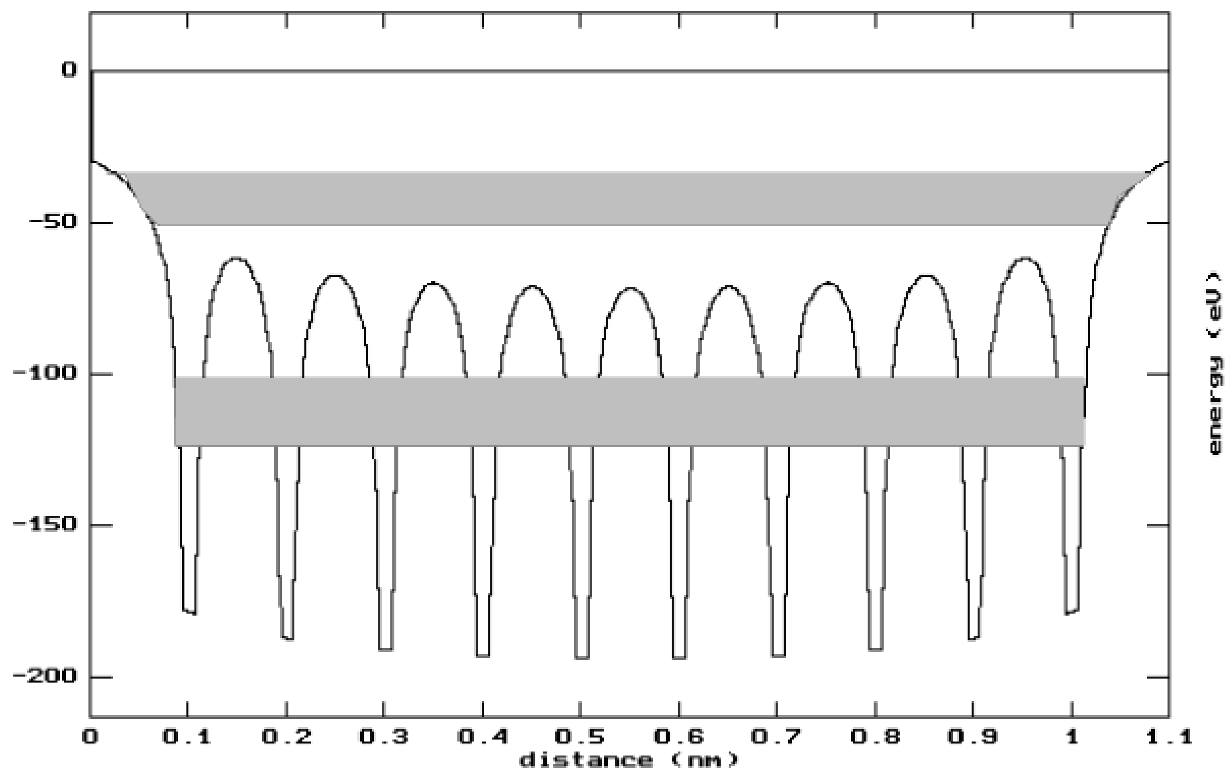
Singh, Belloni and Christian, “Improving Student Understanding of Quantum Mechanics”, Figure 2



Singh, Belloni and Christian, “Improving Student Understanding of Quantum Mechanics”, Figure 3



Singh, Belloni and Christian, "Improving Student Understanding of Quantum Mechanics", Figure 4



Singh, Belloni and Christian, “Improving Student Understanding of Quantum Mechanics”, Figure 5

Open the simulation by double-clicking the green arrow associated with this exercise. Shown is the wave function for an electron in the one-dimensional infinite square well at time $t = 0$ given by $\psi(x,0) = \sqrt{\frac{2}{L}}\sin(5\pi x/L)$ for which in the simulation $\hbar = 2m = L = 1$. Click on the time-development to evolve the wave function in time.

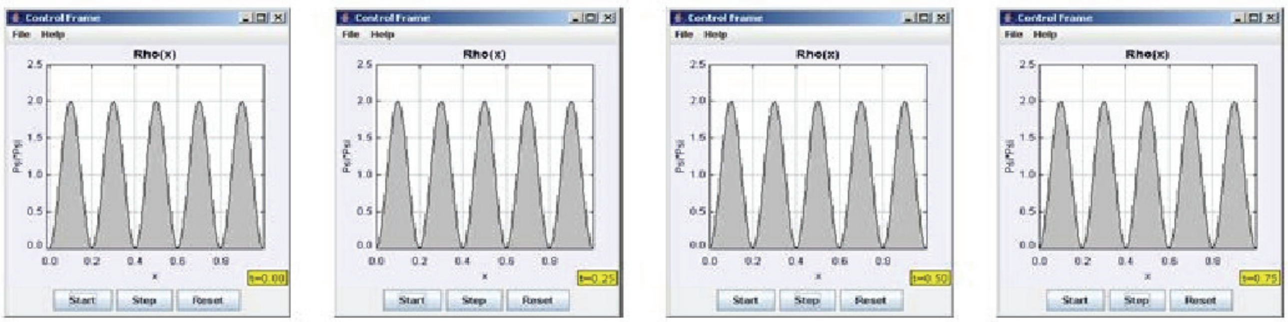
(V) Does $|\psi(x,t)|^2$ for a given x change with time in the simulation?

(a) Yes.

(b) No.

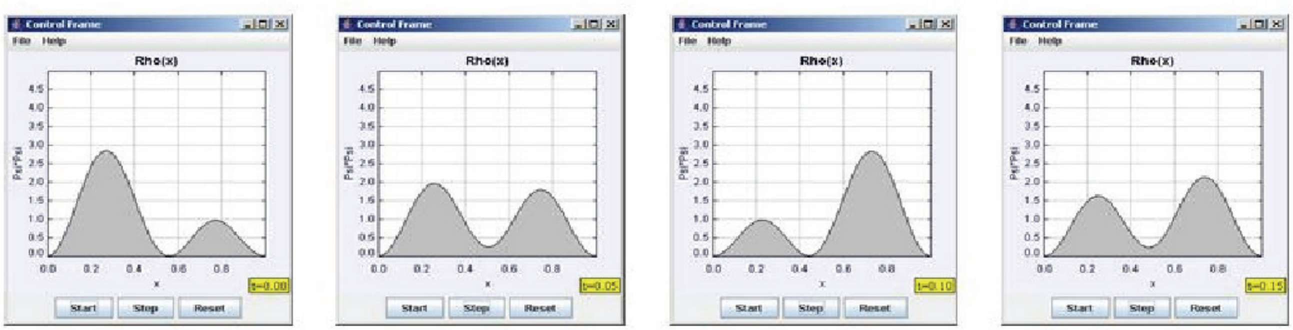
(c) I do not know what I should be looking at in the simulation.

a



3. Now open the simulation (double-click the green arrow) and choose the initial wave function $\Psi(x,0) = \sqrt{\frac{1}{8}}\psi_1(x) + \sqrt{\frac{7}{8}}\psi_2(x)$. Watch the time evolution of $|\Psi(x,t)|^2$. Is the time evolution of this wave function consistent with what you predicted earlier? Explain.

b



Singh, Belloni and Christian, “Improving Student Understanding of Quantum Mechanics”, Figure 6