

# Accounting for Errors in Model Analysis Theory: A Numerical Approach

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By studying the patterns of a group of individuals' responses to a series of multiple-choice questions, researchers can utilize Model Analysis Theory to create a probability distribution of mental models for a student population. The eigenanalysis of this distribution yields information about what mental models the students possess, as well as how consistently they utilize said mental models. Although the theory considers the probabilistic distribution to be fundamental, there exists opportunities for random errors to occur. In this paper we will discuss a numerical approach for mathematically accounting for these random errors. As an example of this methodology, analysis of data obtained from the Lunar Phases Concept Inventory will be presented. Limitations and applicability of this numerical approach will be discussed.

## Introduction

Using model analysis theory, researchers can create a model density matrix, which is approximately a probability distribution of mental models for a student population [1]. The theory then utilizes an eigenanalysis of the class model density matrix to extract information about what mental models the students possess, as well as how consistently they use said mental models. While the theory considers the model density matrix to be fundamental, opportunities for random errors exist, although the error propagation is not dealt with specifically [1].

This paper will investigate how these random errors propagate through the eigenanalysis and affect the final results. To this end we will derive a separate matrix that represents the error values, and then we will use this information to place numerical bounds on the eigenvalues resulting from the eigenanalysis. We will bring the errors into play and observe how high and low the eigenvalues can be. If the bounds are dramatically different from the eigenvalues obtained from the original matrix that did not contain error; therefore, little fluctuations in values will drastically alter the results. If this is indeed the case, then model analysis theory may not have strong predictive power.

To illustrate the entire process we will provide a detailed example using data obtained

from the administration of the Lunar Phases Concept Inventory (LPCI) to a large lecture astronomy course [2]. Finally, we will discuss some limitations of the methods used in this paper to determine the bounds.

## Deriving the Error Matrix

If we let  $p_x, 0 \leq p_x \leq 1$  be the probability that students use model  $x$ , then model analysis represents the model state for a student as [1]

$$v = \begin{pmatrix} \sqrt{p_1} \\ \sqrt{p_2} \\ \vdots \\ \sqrt{p_n} \end{pmatrix}. \quad (1)$$

Since the values of  $p_x$  are the only measured values that are used, we should introduce the random error at this stage. We will let  $\epsilon_x, -1 \leq \epsilon_x \leq 1$ , be the amount that  $p_x$  can be wrong. Thus we get the new model vector,  $z$ :

$$z = \begin{pmatrix} \sqrt{p_1 + \epsilon_1} \\ \sqrt{p_2 + \epsilon_2} \\ \vdots \\ \sqrt{p_n + \epsilon_n} \end{pmatrix}. \quad (2)$$

The single-student model density matrix is defined as [1] d,

$$d_v = vv^t = \begin{pmatrix} p_1 & \sqrt{p_1 p_2} & \dots & \sqrt{p_1 p_n} \\ \sqrt{p_2 p_1} & p_2 & & \sqrt{p_2 p_n} \\ \vdots & & \ddots & \vdots \\ \sqrt{p_n p_1} & \sqrt{p_n p_2} & \dots & p_n \end{pmatrix} \quad (3)$$

and the single-student model density matrix with errors is defined as

$$d_z = zz^t = \begin{pmatrix} p_1 + \epsilon_1 & \sqrt{p_1 p_2 + p_1 \epsilon_2 + p_2 \epsilon_1 + \epsilon_1 \epsilon_2} & \dots & \sqrt{p_1 p_n + p_1 \epsilon_n + p_n \epsilon_1 + \epsilon_1 \epsilon_n} \\ \sqrt{p_2 p_1 + p_1 \epsilon_2 + p_2 \epsilon_1 + \epsilon_1 \epsilon_2} & p_2 + \epsilon_2 & & \sqrt{p_2 p_n + p_2 \epsilon_n + p_n \epsilon_2 + \epsilon_2 \epsilon_n} \\ \vdots & & \ddots & \vdots \\ \sqrt{p_n p_1 + p_1 \epsilon_n + p_n \epsilon_1 + \epsilon_1 \epsilon_n} & & & p_n + \epsilon_n \end{pmatrix} \quad (4)$$

Thus, if we subtract (3) from (4), we will get a matrix that contains the difference between the single-student model density matrix and the single-student model density matrix with errors. In short, this new matrix is the error that could be associated with each single-student model density matrix. The matrix takes the form of

$$e = d_z - d_v = \begin{pmatrix} \epsilon_1 & \sqrt{p_1 p_2 + p_1 \epsilon_2 + p_2 \epsilon_1 + \epsilon_1 \epsilon_2} - \sqrt{p_1 p_2} & \dots & \sqrt{p_1 p_n + p_1 \epsilon_n + p_n \epsilon_1 + \epsilon_1 \epsilon_n} - \sqrt{p_1 p_n} \\ \sqrt{p_2 p_1 + p_1 \epsilon_2 + p_2 \epsilon_1 + \epsilon_1 \epsilon_2} - \sqrt{p_2 p_1} & \epsilon_2 & & \sqrt{p_2 p_n + p_2 \epsilon_n + p_n \epsilon_2 + \epsilon_2 \epsilon_n} - \sqrt{p_2 p_n} \\ \vdots & & \ddots & \vdots \\ \sqrt{p_n p_1 + p_1 \epsilon_n + p_n \epsilon_1 + \epsilon_1 \epsilon_n} - \sqrt{p_n p_1} & & & \epsilon_n \end{pmatrix} \quad (5)$$

The single-student model density matrices are averaged to create a class model density matrix. We must assume that averaging the student error matrices will result in a matrix that is identical, or at least very similar to, the matrix one would get by placing the classes data in  $e$ . Thus we can use  $e$  with the entire class data, as the error matrix for the class density matrix. This is crucial, since we need to perform the eigenanalysis on the class density matrix, thus needing a corresponding error matrix.

### Using the Error Matrix to Find the Bounds

Since each value of  $\epsilon$  can be positive or negative, for simplicity sake, we let each value in  $e$  be both positive and negative. It is clear that when all of the epsilon values are positive, the error matrix will assume the positive values; however, when some values for epsilon are negative, values for entries in the error matrix can also be negative. Different combinations of epsilons being pluses and minuses can create many values for the entries in the error matrix, but we will only worry about  $\pm e_{ij}$ , as all other

possible values will exist in this range, and it makes calculations much less tedious.

Because each value in  $e$  can be either plus or minus, we must check all possible combinations of pluses and minuses for the error matrix. However, some combinations can be ruled out right away simply by observing the nature of the error matrix. It is clear that the error matrix is symmetric, so the pluses and minuses must also be symmetric, so

$$\begin{pmatrix} + & - & - \\ - & + & + \\ - & + & - \end{pmatrix}, \text{ or } \begin{pmatrix} + & + & + \\ + & - & - \\ + & - & + \end{pmatrix},$$

are examples of combinations of pluses and minuses in the error matrix that we need to check. However we do not need to check

$$\begin{pmatrix} + & + & - \\ - & + & + \\ - & + & - \end{pmatrix}, \text{ or } \begin{pmatrix} + & + & + \\ + & - & - \\ - & - & + \end{pmatrix}.$$

For every legitimate combination we take the class model density matrix without errors and add the error matrix with the plus and minus coefficients. Once they are added, we find the eigenvalues. Over all of the possible combinations we observe how large and small each individual eigenvalue can be. These smallest and largest values become our bounds, with respect to each eigenvalue.

Since having many different epsilon values make the error matrix more complicated, and since we are going to have to assume values

for epsilon, we will take  $\varepsilon = \max\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$ .

This creates a new error matrix

Thus the values in  $e$  are greater than the “true” error values. The error bounds this method produces will be just that, upper bounds, and the actual error in the eigenvalues will probably be less than the results of these calculations.

### An Example

For the example we will use pretest data from the administration of the LPCI to a large-lecture astronomy course of 194 students. The LPCI is a 20-item multiple-choice instrument designed to use model analysis theory to investigate students models of lunar phases over eight different concept dimensions. In this example we will only look at one concept dimension, specifically the dimension that measures students understanding of orbital period.

The class model density matrix is

$$\begin{pmatrix} .42 & .11 & .05 \\ .11 & .42 & .06 \\ .05 & .06 & .16 \end{pmatrix}. \quad (6)$$

Values for epsilon are unknown quantities at this time, so for the purpose of this example we will choose an epsilon of .05. This means that all the probabilities are off by no more than 5%. This epsilon yields an error matrix

$$\begin{pmatrix} .05 & .05 & .0549 \\ .05 & .05 & .0549 \\ .0549 & .0549 & .05 \end{pmatrix}. \quad (7)$$

Checking all combinations, we obtain the following upper and lower bounds on the eigenvalues.

lower bound	no error	Upper bound
0.43	0.5457	0.6813
0.2101	0.3103	0.4327
0.0467	0.144	0.2099
Table 1		

This table is only for a singular value of epsilon. To obtain a better feel for how epsilon affects the eigenvalues we should complete the calculations with a set of different epsilon values. If we did this for many values and tried to create tables like the one containing the bounds it would be difficult to see what was happening due to the large volume of numbers.

Instead we isolate an eigenvalue (i.e. the largest, smallest, or middle one) and observe the maximum difference between the lower bound or upper bound and the value without error. This number is simply how far off the eigenvalue can be. We can look at this number for many values of epsilon and then produce a plot of epsilon versus the difference in the eigenvalues.

Figure 1 was created using the orbital period pretest data, isolating the largest eigenvalue for each value of epsilon. Figure 1 tells us how much the eigenvalue of .5457 can be off for different values of epsilon.

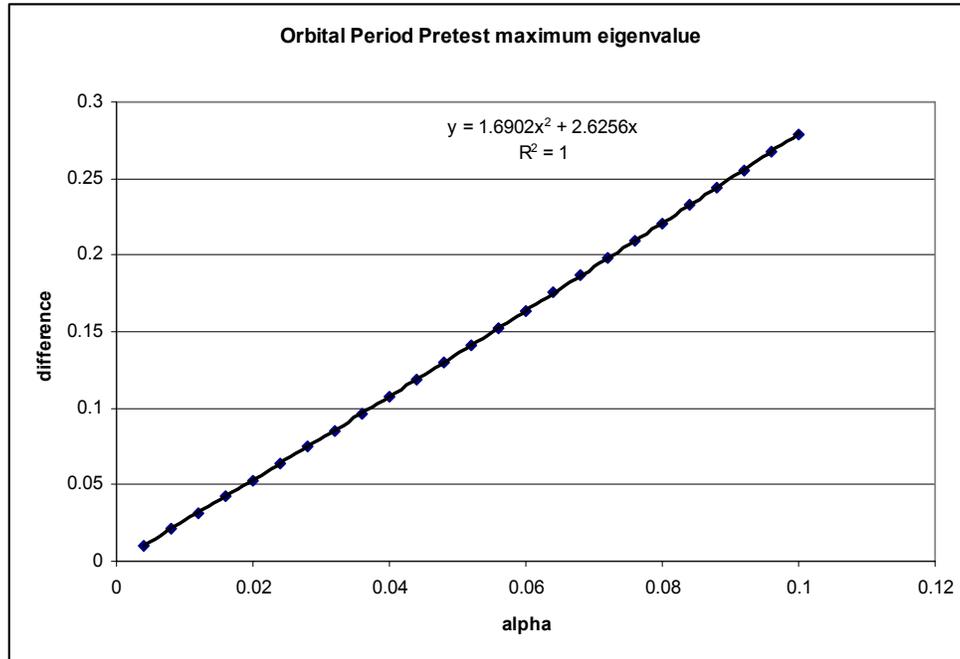


Figure 1. Epsilon versus the difference in the maximum eigenvalue

For example, if the initial probabilities are off by 10% the eigenvalue is off by no more than .28. Similar curves can be generated for the each of the eigenvalues.

### Limitations

Since each legitimate combination must be checked, this process takes time. Even though all of the data used in the example was generated via a computer program written in MATLAB, it still took time to find all of the data points for the curve. As the size of the matrix grows, the time needed to check all of the combinations will also increase, since there are more combinations with larger matrices. Specifically, an  $n \times n$  matrix has  $2^{\frac{n(n+1)}{2}}$  combinations that must be checked. This numerical method will only be practical for small matrices

### Conclusion

Using the methods discussed in this paper researchers can easily determine how stable model analysis theory is. Stability is essential since there are places where random errors can

occur. With stability the researcher can use the results he or she gets from the eigenanalysis without fear of those results being that far off with a little fluctuation in values due to random error.

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### References

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