

# Faculty Discourse in the Classroom: Meaning in Mathematical Moves

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**Abstract:** We analyze the discourse of faculty presenting derivations in which they manipulate mathematical equations to illuminate a physical principle. Observations are interpreted through a lens of *symbolic forms*, conceptual and contextual meanings that are embedded in the equation. When an equation is manipulated (e.g. bringing terms to one side or another), different forms are emphasized, changing the meaning of the equation. We argue that this framework can make explicit the faculty motivations for the moves, and present two observations of manipulations that appear to have distinctly different reasons. The first manipulation brings about a change in context from a physics to a mathematical frame. In the second, a thematic manipulation --- grouping all terms of a common variable --- reveals an important conceptual point about a driven harmonic oscillator. While there is direct evidence from the observed faculty to support the inference of motivation, in neither case is the reasoning made clear to the students. The study of discourse represents a new direction in which physics education researchers can study and inform the classroom.

**Keywords:** Physics Education Research, Faculty Discourse

**PACS:** 01.40.-3, 01.40.gb

## INTRODUCTION

Physics embeds conceptual meaning within a mathematical formalism. Nevertheless, physics education research has historically separated concepts from equations. Early research presented ample evidence that students' ability to manipulate equations or solve traditional physics problems did not translate to conceptual understanding.<sup>1</sup> As a result, curriculum development<sup>2-3</sup> and assessment<sup>4</sup> efforts focused on the conceptual understanding, often separating the math from the concepts. Research on quantitative problem solving emphasized conceptual understanding either at the beginning of the process, to determine which equations to use, or at the end, to check for plausibility and interpret an answer's physical meaning.<sup>5-6</sup> Kuo et al.<sup>7</sup> analyzed over a decade's worth of papers from eight leading journals in physics and science education and found "*no studies that focused upon the mathematical processing step in quantitative problem solving or described alternatives to using equations as computational tools.*" The separation of concept from math puzzled at least one of the author's colleagues, a dedicated teacher who said quite earnestly: "*The concepts are the math and the math are the concepts.*"

### Meaning in Math: The Symbolic Form

The premises of this work are (1) physics equations contain deep conceptual and contextual meaning, (2) rearranging an equation can change this meaning, and (3) manipulations have distinct motivations that are rarely presented to the student.

In order to articulate the conceptual and contextual meaning embedded in mathematical equations, Sherin<sup>8</sup> defined *symbolic forms* as an elemental relation associating a simple conceptual schema with a pattern of symbols. It is critically important to note the form's context-dependence. Two equations that are mathematically identical may have very different physics schema. Consider the following two equations:

$$\vec{v}_f = \vec{v}_0 + \vec{a}t \quad (1)$$

$$\vec{F}_{net} = m\vec{g} + k\vec{x} \quad (2)$$

Mathematically, Eqs. 1 and 2 both equate a quantity with the sum of two separate terms. Physicists add conceptual context, recognizing Eq. 1 as a kinematics equation and Eq. 2 as a Newton's 2<sup>nd</sup> Law equation involving gravitational and Hookian forces. This gives a different meaning to topologically equivalent math. The final velocity  $v_f$  in Eq. 1 is composed of an initial value  $v_0$  and changes due to an acceleration  $a$ . Sherin refers to this as the "base+change" form. (Eq. 1 also defines the acceleration. That this is clarified by re-writing to  $\vec{a} = (\vec{v}_f - \vec{v}_0)/t$  supports our first premise.)

The concepts of "base" and "change" are not relevant in Eq. 2. Instead, the net force  $F_{net}$  is comprised of two separate parts, each of which is a force in its own right. Sherin labels this the "sum of the parts" form. One does not think of either force as representing a "change" in the net force, and so the base+change form does not apply.

As faculty, we want students to recognize and appreciate these associations, which represent knowledge above and beyond rote manipulation of

symbols. Evidence for this is seen in the careful analysis of how faculty use mathematical formalism.

### Frames

Physics faculty present mathematical formalism in a number of different manners or “frames.” A frame<sup>9</sup> is the contextual surrounding of an interaction, and understanding (most often subconsciously) the active frame is necessary for effective communication. A playful example is a monkey’s response to another’s bite, which depends on whether they are in a playful or antagonistic frame. Similar examples are seen in the different communication styles between close friends and new acquaintances.<sup>10</sup> Frames are fluid, changing during the interaction (c.f. Tannen and Wallat<sup>11</sup>).

We have observed at least two distinct frames in which physics faculty present derivations. The first, which we will not discuss in detail, is termed the “just math” frame. It is often marked by pejorative faculty statements, for example “now, this is *just* math” or “from here, we *simply* turn the crank” or “this is *just algebra*.” The faculty frames the math as containing little physics or pedagogical value, and that they are being included simply for completeness.

Describing the second frame is the primary purpose of this paper. This frame consists of mathematical derivations and/or manipulations that the instructor believes are important to convey some level of understanding. In interviews, faculty struggled to articulate a precise motivation for presenting derivations but maintained a strong belief of its value in conveying *something* important.

### METHODS

We undertake a qualitative case study with an emergent analysis that, due to the idiosyncrasies of discourse, does not employ a formal rubric. (This approach mirrors that of Tannen.<sup>9</sup>) To maintain rigor in the absence of a formal rubric, we follow the process for qualitative research outlined by Creswell<sup>12</sup> and Otero and Harwell,<sup>13</sup> consisting of:

1. *Description*
2. *Analysis for Themes*
3. *Confirmation/Validation*

The description is presented without interpretation and is intended to convey the environment and observed behavior. Analysis is purely interpretive, and strives to be both plausible and evocative. To ensure plausibility, analyses were presented to a group of five discipline-based education researchers. Only themes that all agreed were logical and plausible were pursued. Confirmatory or validating evidence was found either in a post-observation interview or in a closer parsing of the verbal discourse. This latter is distinctly different

from the parsing of the *mathematical* discourse that comprises the bulk of the analysis.

We note our observer biases. The researchers are physicists with knowledge of the subject, and thus assume meaning behind the math. To compensate, all observations were discussed with faculty from departments of chemistry and biology during analysis.

### Classroom Context

Both observations were made in a junior level Classical Mechanics course. The class met in a traditional lecture-type classroom, with students sitting in rows facing the front. The instructor was an experienced Senior Lecturer with 10+ years of teaching at all levels of the physics curriculum. He had taught the course the previous academic year and consistently received excellent evaluations. The observed lecture was on resonance in the driven harmonic oscillator and was performed with minimal student interaction save for occasional direct questions from the instructor addressed to the entire class.

We produce below, without annotation, the series of mathematical equations the instructor writes on the board in order to move from Newton’s 2<sup>nd</sup> Law to an expression for the amplitude of the damped oscillator:

$$m\ddot{x} = -kx - c\dot{x} + F(t) \quad (1)$$

$$m\ddot{x} + kx + c\dot{x} = F(t) \quad (2)$$

$$F(t) = F_0 \cos \omega t, c = 0 \quad (3)$$

$$m\ddot{x} + kx = F(t) \quad (4)$$

$$x = A \cos \omega t + \varphi \quad (5)$$

$$-m\omega^2 A \cos(\omega t + \varphi) + A \cos(\omega t + \varphi) = F \cos \omega t \quad (6)$$

$$A(k - m\omega^2) = \frac{F \cos \omega t}{\cos(\omega t + \varphi)} \quad (7)$$

$$A = \frac{F}{k - m\omega^2} \quad (8)$$

There was very little instructor discourse during the derivation, with the instructor often simply verbalizing aloud verbatim the math that he is writing, e.g. (phonetically) “*em ex double-dot equals minus kay ex minus see ex dot plus eff of tee.*” This behavior was observed in multiple classes across several instructors.

### MANIPULATION TO SET CLASSROOM FRAME

Our first observation centers on the move from Eq. 1 to Eq. 2. We describe the observation, interpret the equations in the context of their symbolic forms, and argue that the rearrangement has changed the frame from one emphasizing physics concepts to one that suggests mathematical approaches.

### Description & Analysis of Themes

After some brief introductory talk reminding students that in the previous class they had discussed damped oscillators, the instructor moves on to say that they were going to add a time-dependent driving force  $F(t)$  and writes, without comment:

$$m\ddot{x} = -kx - c\dot{x} + F(t) \quad (1)$$

He then says “Now, of course, we can write it, you know” and writes

$$m\ddot{x} + c\dot{x} + kx = F(t) \quad (2)$$

He pauses for a brief moment (<3 seconds), during which time no students interrupt for clarification or ask questions. He then continues with his derivation.

To unpack the meaning and reasoning behind this move we look at the symbolic forms and concepts how they are affected by the reorganization. Eq. 1 contains the “parts of a whole” form, with the various forces grouped together on one side. Specific meanings are associated with the two “-“ signs. The first indicates a restoring force that opposes displacement, while the second indicates damping. These are fundamentally different (from a physics perspective); consider the dissonance when one associates the language of one with the other:  $-kx$  cannot be thought of as a “damping” nor  $-c\dot{x}$  in any way “restorative.”

This meaning is obscured in Eq. 2. One can no longer talk about a net force, since the forces are now divided across the equals sign, with some of the forces now being added to a term,  $m\ddot{x}$ , that, while having the units of force is rarely directly called “a force.” By leaving the  $F(t)$  term alone, the instructor now explicitly treats some forces differently than others, and the idea of a net force is no longer emphasized. Similarly, the concepts of restoring and damping forces are now hidden, since the “-“ signs have disappeared. We do not argue that these concepts have vanished completely, but they are no longer emphasized.

What exists in its place, however, is an equation that, while lacking its original physics conceptual meaning, strongly suggests a mathematical solution. By grouping all terms involving the variable  $x$  (and its derivatives) on one side and the only term solely dependent on time on the other, the instructor has put the equation into a standard form for an inhomogeneous ordinary differential equation. He thus changes the contextual frame of the class from one in which physics concepts are paramount to one that involves an application of a technique: how to find solutions to ordinary differential equations. The instructor assumes his students are familiar with this approach, and they can employ their own knowledge schema and fill in steps as the instructor writes the solution on the board. Indeed, there are no questions asked nor other expressions of bewilderment, so, while

not providing definitive proof, it is suggestive that the change in frame is not unfamiliar to the students.

### Confirmation

A followup interview with the instructor was conducted in which he was shown the video and asked his motivation in the first rearrangement of the equation. He stated explicitly that the equation was changed in order to set up the problem to be solved as a differential equation, confirming our hypothesis of an intentional frame change. While we suspect the students intuitively sensed the change, student interviews are needed to confirm this. A productive discussion might ensue about making such frame changes explicit by addressing the motivation behind the mathematical manipulations (e.g. why was the term  $F(t)$  treated differently than the other forces.)

## MANIPULATION TO REVEAL AN IDEA

We now focus on the mathematical move that takes the derivation from Eq. 6 to Eq. 7.

### Description & Analysis of Themes

The instructor, after reaching Eq. 6:

$$-m\omega^2 A \cos(\omega t + \varphi) + A \cos(\omega t + \varphi) = F \cos \omega t \quad (6)$$

says: “Ok. Now, we can simplify this some. We can say that” and writes Eq. 7

$$A(k - m\omega^2) = \frac{F \cos \omega t}{\cos(\omega t + \varphi)} \quad (7)$$

As before, he verbalizes the variables as he writes.

No variables or terms have disappeared, and so the move is purely reorganizational. No motivation beyond the desire to simplify is given, nor any reason into why this reorganization is to be preferred over the other. In particular, why the term  $(k - m\omega^2)$  remains on the left side of the equation is never explained.

The purpose of this reorganization is to make explicit a compound symbolic form, combining elements of equivalence (if one side is time independent, then the other side must be as well) with aspects of cancellation (time dependence in the numerator must be canceled by equivalent time dependence in the denominator). These ideas are not present in Eq. 6, since all terms are time dependent. While the instructor could still argue his mathematical point (that  $\varphi = 0$  or  $\pi/2$ ) that the time dependence of all terms must be identical, he chooses to reorganize, implying some value in the reorganization itself.

We argue that this is a thematic reorganization, grouping together all terms that depend on a variable (in this case time). This requires a sophisticated interpretation of symbols; for the moment, time-dependence is the most important characteristic. Similar moves are seen in the method of separation of variables when applied, for example, to solving the heat or wave equation. Whereas that motivation is emphasized when covered in the Mathematical Methods course, here it is not even mentioned. We note that he delays the mathematical step of factoring out  $(k - m\omega^2)$ , as this might cloud the issue of time-dependence, even though doing so adds an extra mathematical step.

### Confirmation

Confirmation for the notion of compound symbolic form is found in discourse analysis of the instructor's comments immediately following the writing of Eq. 7. He says (emphasis in bold and annotation in brackets):

**Right.** *So if we do that, this [left side] is obviously some type of constant. It is not dependent on time. Which means that in order for this [the equation] to be a solution, this [right side] can also not be a function of time.* **Alright.** *So that means you've got two possible solutions. Either you have phi equals zero or phi equals pi. And if phi equals zero, this ratio equals one and if phi equals pi this ratio equals minus one.* **Right.** *So that's the only two solutions you can get.*

We claim that the instructor implicitly or explicitly recognizes the compound nature of the concept he is teaching and uses verbal cues to delineate the components. The first concept is that of equivalence, and is bracketed by the terms "Right" and "Alright." The text between the terms "Alright" and "Right" represent the second concept, that of cancellation in a proportionality. The words such as "Right" and "Alright" therefore serve as contextual markers to set off the two individual concepts that together make up the meaning of Eq. 7. Usage is most likely subconscious; only after this study did one author (SVF) became aware of his tendency to similar terms.

### CONCLUSIONS

We have presented two relatively simple mathematical moves made during a single derivation and argued for different underlying motivations. The first changed the contextual frame of the class environment, whereas the second served to emphasize a compound concept. Each is made possible by the existence of distinct meaning in the equations, meaning that is changed when the equation is reorganized. Reorganization can serve to obscure physical context

so as to emphasize a mathematical solution, such as when forces are split across an equation to group together terms of a single variable. It can also isolate specific behaviors, such as when time-dependent terms are grouped together. We do not believe these findings are controversial; physics faculty interviewed readily agree to the concept of symbolic form and meaning. Nevertheless, the articulation of the forms and focus on the meaning brings a new dimension to lectures involving derivations. In particular, it raises the possibility of engaging students in discussions on conceptual and contextual meaning of mathematical equations in physics. This richer representation is closer to how physicists see and use equations, and could lead to a more complete understanding by the students.

### ACKNOWLEDGMENTS

We gratefully acknowledge members of RIT's Science & Mathematics Education Research Collaborative (SMERC) for many fruitful discussions.

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