Nesting In Graphical Representations In Physics

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Abstract. We develop a theoretical model for understanding one way, "nesting," that space is used in graphics from within and outside physics. Nesting can be used to increase a graphic's capacity for displaying several dimensions of information, beyond the two dimensions afforded by a flat page. We use the model of nesting to analyze previously observed student difficulties with electromagnetic waves, to predict how physics students would interact with certain graphics, and to generate new multivariate graphics in physics for instruction and for research on student thinking. Finally we apply the nesting model to explain the multidimensionality of certain kinds of gestures in physics education.

Keywords: graphics, spatial cognition, gesture.
PACS: 01.40.Fk, 01.40.Ha

INTRODUCTION

Spatial cognition is abundant in physics, in two senses: space and literal movement of objects in and through it are explicit subjects of physics; and the space of our bodies, papers, and imaginations is used to describe spatially that which is not literally spatial. For instance, an electric field exists at many points in space, but the electric field at any one point is not *itself* extended in space, though it is often depicted with a drawn arrow, with *longer* used to mean *more*, as though it were. Yet, we have so many things that we want to use space to describe that it is quite easy to "run out of space." So, how can we use limited space to communicate such a wide variety of ideas, often in complex relation to each other? We develop the theoretical structure of *nesting* to explain one way in which people manage many ideas in limited space.

Example Of Nesting In Physics: Energy Level Diagrams In Quantum Mechanics

The diagram in Figure 1 shows the energy eigenvalues and eigenfunctions for a particle in an infinite one-dimensional square well potential. The eigenvalues and eigenfunctions are presented together in a hybrid graphic, in which the vertical dimension is doubly-occupied with meaning. That is, two abstract physical quantities are represented with the vertical dimension of the graphic, but in different ways. One vertical coordinate is the quantum wave-function amplitude (which, for this system, has dimensions of $1/\sqrt{(length)}$) and is repeated five times. The other vertical coordinate is the energy eigenvalue of each function. Though the meaning of the diagram is not obvious, we claim that it is also inherently not pedagogically problematic, despite the apparent possibility of confusion about the interpretation of the vertical coordinate. We describe the relationship between these two vertical coordinates as one of nesting. In a nested diagram, space has one meaning within the nested region and another meaning outside it. If these different meanings are quantified coordinates, we call coordinates "inner" and "outer" coordinates. In the diagram in Fig. 1, the wave function amplitude is the inner vertical coordinate, and the energy is the outer vertical coordinate.

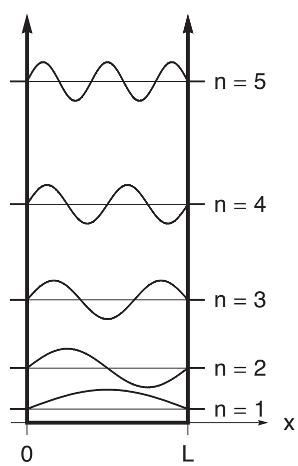

FIGURE 1. A diagram depicting the five lowest energy eigenvalues and their eigenfunctions for a particle in a one-dimensional infinite square well. The diagram involves nested coordinates, with energy as the outer vertical coordinate, and the amplitude of the quantum wave function as the inner vertical coordinate. Figure reproduced from Ref. 2 with permission.

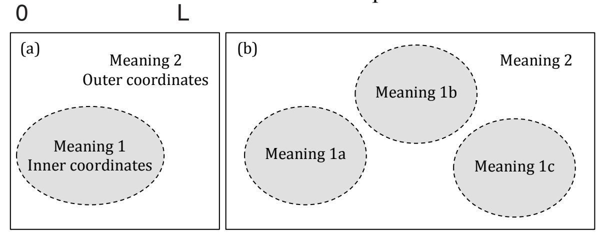

FIGURE 2. (a) Generic nesting diagram. The meaning attributed to location within the bounded space is of a different character from that attributed to location outside the bounded space. (b) "Small multiples," in which the meaning of location on the inside of several bounded spaces is analogous.

In Fig. 1, the inner coordinate is repeated five times; this repetition of graphical meaning is called using "small multiples," as explained in Ref. 1. When a diagram uses small multiples, the effort the viewer puts in to interpreting the small graphics is spent only once, since they all follow the same rules of interpretation. The ideas of nesting and small multiples are depicted in the abstract in Figure 2.

Examples Of Nesting In Everyday Life

There are many examples of nesting in the everyday use of space. A road atlas will often have on one page both a zoomed-out representation of a state, with insets that show a zoomed-in view of the major cities in that state. These insets are placed with no regular coordination between the inner and outer spaces. An analog speedometer is an example that is more abstract than a map, since it represents speed (which is not location) with an angular location. The meaning in that space, where "more clockwise" means more speed, is limited to the speedometer space. Though the varying meaning of space within a field of view may seem like it would cause cognitive trouble, its common use suggests otherwise.

PRINCIPLES OF GRAPHICAL EXCELLENCE

We build our ideas about nesting in graphics on a model for graphical excellence as articulated by E. Tufte [1]. Tufte's model for graphical excellence emphasizes a quality of efficiency, which "gives to the viewer the greatest number of ideas in the shortest time with the least ink in the smallest space."

Advantages of Multivariate Graphics

Communicating the "greatest number of ideas" means telling as much as possible about the co-variation and causal relationships between variables. According to Tufte, (Ref. 1, p. 129) graphics tell a story of reality that is itself inevitably multivariate. In order to be faithful to the richly detailed relationships among data and models, Tufte explains, graphics should be multivariate because a multivariate graphic promotes multivariate reasoning. Inversely, we understand Tufte's model to predict that a two-dimensional graphic of a more complex multivariate network of relationships would mislead the user to reason mostly with only two variables. Such phenomena may be observable, for example, in situations with students reasoning about PV diagrams in thermodynamics, which often leave much of the complex relationships among thermodynamic quantities graphically unrepresented.

Nesting is one strategic way to increase the dimensionality of information presented graphically.

NESTING IN VECTOR FIELD DIAGRAMS

Vector fields (*e.g.,* electric field, magnetic field, velocity field of a fluid) are commonly represented with an "array-of-arrows" graphic format. This format involves graphic space that is occupied with multiple meanings; each point on the graphic represents a physical location, but arrows drawn on the diagram do not point from one location to another – they point from a local origin in electric-field space, *e.g.,* to a point in that space that represents a particular vector value of E-field. Because of this "double occupancy," the diagram presents the possibility of confusion for a beginner: Does the arrow point from one location to another, or not? And if not, to what does it point? However, the presence of many examples of nested spaces in life outside of school suggests that the nesting itself is not problematic, though a student's understanding of electric field may be.

The nesting model suggests that learners will have significantly less trouble interpreting the inner meaning as distinct from the outer meaning, as long as the nesting schema is cued, *i.e.*, the representation cues the learner to expect transition boundary (or region, more likely) of meaning, to estimate where that transition is, and to guess what the meanings are. The model also suggests that nested diagrams can become problematic if the partition of meaning is violated. For example, in an electric field vector diagram, how close does the arrow representing the electric field at one location come to another location? A gradual change in the diagram towards "intersecting ink" should cause viewers difficulty in interpreting the diagram. How close is too close? And when it is too close, what goes wrong?

EXPLAINING STUDENT PROBLEMS WITH REPRESENTATIONS

The nesting model can help us understand certain previously observed phenomena of student thinking. For example, as reported in Ambrose et al. (1999) [3], students had major difficulty interpreting a particular graphic depiction of an electromagnetic (EM) plane wave (see Fig. 1 of Ref. 3). The graphic emphasizes the sinusoidal dependence of the fields on the coordinate x, along which the wave propagates, but hides other aspects of the wave, *viz.*, the uniformity of the field in each plane perpendicular to the direction of propagation, and thus the physical *in*-significance of the placement of the x-axis itself. The graph also fails to alert the viewer to its own double occupancy, in which spatial extent represents both field quantity and location. By extending the field vector arrows through various field locations (*i.e.*, not restricting the field meaning to "inside the nest"), the graphic "anti-cues" the nesting schema and leads the viewer to interpret a single meaning for spatial extent within the graphic. The visual salience of point P (in Ref. 3), as being the one point "outside the wave," seems to communicate something, though when it is correctly interpreted, it represents nothing.

We understand the following statement by Ambrose *et al.* to be consistent with our explanation of the diagram in terms of nesting: "The most common error has been to ascribe to the plane EM wave a finite spatial extent in the plane perpendicular to the direction of propagation." (p. 892 of Ref. 3). The authors also claim that some difficulties with EM waves transcend the diagram. This claim could be tested by repeating the interviews performed in that study with a different diagram of an EM plane wave, such as that found on p. 991 of Ref. 4. The diagram shows a series of parallel planes in perspective. Each plane, which is perpendicular to the axis of propagation, contains a regular, sampled array of marked field locations. Both $\vec{E}$ and $\vec{B}$ field vectors are shown at each location in the plane. The diagram shows that the fields are uniform within each plane, and vary (perhaps sinusoidally) from one plane to the next. The diagram uses nesting (though not explicitly, since it predates the present theoretical development) because the field vectors have spatial extent in the graphic, but their extent is less than the spacing between the marked field locations. We predict that interviews based on this "shish kebab" diagram would reveal much less difficulty of interpretation for students.

EXPLICITLY USING NESTING TO CREATE NEW GRAPHICS

Example 1: Complex Scalar Plane Wave

The complex scalar function $\psi(x) = Ae^{ikx}$ is usually graphed by separating the function into real ($A\cos kx$) and imaginary ($A\sin kx$) parts, and plotting those on a single set of axes, as functions of the position x. The vertical axis is thus doubly-occupied with meaning; the two meanings are usually distinguished by graphing the real and imaginary parts with different colors or textures. Various features of this representation can be understood as problematic: the constant magnitude of the function is obscured; the relationship between the real and imaginary parts appears to be one primarily of translation symmetry rather than of interdependent mutually orthogonal projections; and the graph bears little resemblance to the most familiar depiction of complex numbers as located in the complex plane.

In contrast, consider a graph using nesting, as shown in Figure 3. In this graph, the magnitude $|\psi|$ is plotted as the outer vertical coordinate, as a function of the outer horizontal coordinate x.

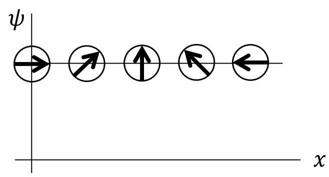

FIGURE 3. Graphic showing a complex scalar plane wave, using the idea of nesting.

The inner coordinates are small multiples of a miniature complex plane that depicts kx as a phase angle. The "mini complex planes" are circled to emphasize their nested meaning.

This nested graphic has the advantages of showing that the function has a constant magnitude, hinting at a mechanism (rotation) for the similar sinusoidal projections, and relating visually in some way to the complex plane. We do not claim yet that the nested graphic is better overall for students; we claim only that nesting as a design strategy can provide access to some fundamental insight into the function that is different from that afforded by the traditional graph.

One Relationship Of Nesting To Use Of Color

A dynamic animation similar to the static graphic in Fig. 3 can be seen in the PhET simulation "Quantum Tunneling and Wave Packets." [5] In the simulation, the complex phase of an incident plane wave can be shown in color. Since color itself is not spatial information, the phase is not presented explicitly as an inside coordinate in a nested use of space. However, the colors cycle through red, yellow, green, cyan, blue, and magenta, so they can be arranged in an angular cycle around the origin of the complex plane. Further play with the simulation reveals how specific colors correspond to specific phase angles. In this sense, the colors "hyperlink" spatial structure of the complex plane to the graph of the wave function. Fig. 3, in which the nesting is explicit, can be understood as communicating the same information, but without the mediating color field. Thus, the explicitly nested diagram can be used as a way to imitate the simulation in writing when color is not available. This convenience is in addition to whatever direct cognitive value it may afford.

Example 2: Spin ½ Quantum States

Representing quantum states of spin-1/2 systems graphically presents a unique challenge, since they appear to require more dimensions of space than we have available, especially if we restrict our representation to the page. The general state can be written as $|\psi\rangle = c_1|+z\rangle + c_2|-z\rangle$, if we use spin-up and spin-down in the z-direction as basis kets for constructing the state. The coefficients c_1 and c_2 are complex numbers, so the general state lives in a Hilbert space of dimensionality C^2, in which a point is specified by *four* real numbers.

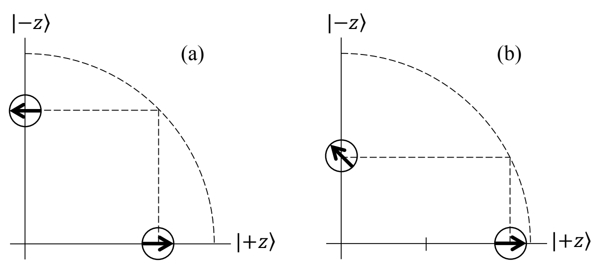

FIGURE 4. Two examples of a new graphic for quantum states for spin-1/2 systems in the +z/-z basis, using the idea of nesting. The large circular arc indicates that the states are normalized. The outer coordinates are the magnitudes of the coefficients for the +z and –z components of the state, while the inner coordinates are the complex phases of those coefficients. (a) $|\psi\rangle = \frac{e^{i\cdot 0}}{\sqrt{2}}|+z\rangle + \frac{e^{i\pi}}{\sqrt{2}}|-z\rangle = \frac{1}{\sqrt{2}}(|+z\rangle - |-z\rangle) = |-x\rangle$ (b) An exercise for the reader to check understanding of the graphic.

The graphic in figure 4 uses the idea of nested coordinates, with an inner coordinate (depicted with an arrow inside a circle) representing the phase of each complex coefficient c_i, and an outer coordinate representing the magnitudes $|c_i|$. The dashed quarter-circle expresses the normalization of the state and thus the relationship between $|c_1|$ and $|c_2|$. The diagram captures all the information contained in a mathematical description of a general state, so any spin-1/2 state could be depicted with it. The diagram can also be extended to spin-1 systems if a third axis is introduced, and the quarter-circle is replaced with an eighth of a sphere.

APPLICATION OF NESTING TO GESTURE RESEARCH

Previous work in gesture research shows that learners sometimes use gesture not in isolated instances but in coordination in space and time. Yoon *et al.* [6] showed learners gesturing in the context of a "mathematical gesture space," where gestures derive part of their meaning through coordination with other gestures. The nesting idea suggests that two gestures, or two parts of a gesture, may function together in a nested relationship, with one gesture acting with an inner meaning and another one with an outer meaning. The "smaller" inner meaning may be expressed with the fingers while the "larger" outer meaning is expressed with the arm, or alternatively, scaling up, with the arms and the whole body, respectively. The fingers/arm case is perfectly illustrated by Chase and Wittmann [7] in the analysis of a video of an interview, in which a student uses the position of his hand to indicate the location of a projectile, while the span between his thumb and forefinger represent the projectile's velocity. The authors argue that the apparent conflict of meaning in the two gestures is complementary rather than inconsistent, and that the student uses different aspects of his body to convey multiple dimensions of information. Thus the nesting framework provides interpretive structure for analyzing simultaneous or sequential gestures of different size and with different meaning.

CONCLUSION

We have introduced nesting as a theoretical model for analyzing how space is used to manage meaning in thinking and communication in a particular way that increases the space's effective dimensionality. A variety of examples of nesting suggest that the strategic use of nesting in the design of new physics graphics and in the impromptu use of the human body during instruction could improve student learning and reveal through empirical studies a new kind of student cognitive competence.

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