

Students' Understanding of the Addition of Angular Momentum

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Abstract. We describe the difficulties advanced undergraduate and graduate students have with concepts related to the addition of angular momentum. We also describe the development and implementation of a research-based learning tool, a Quantum Interactive Learning Tutorial (QuILT), to reduce these difficulties. The preliminary evaluation shows that the QuILT on the addition of angular momentum is helpful in improving students' understanding of these concepts.

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INTRODUCTION

There have been many investigations of students' difficulties in learning quantum mechanics (QM) [1-4]. Here, we focus on investigation to identify student difficulties with the addition of angular momentum in QM. The investigation involved administering written tests in various QM classes and conducting in-depth individual interviews with a subset of undergraduate and graduate students at the University of Pittsburgh (Pitt) and other universities.

Based upon the insight gained from the investigation of difficulties, we developed a Quantum Interactive Learning Tutorial (QuILT) [5] to help students better understand the formalism of the addition of angular momentum in QM. We briefly summarize the development and evaluation of the research-based QuILT. After the preliminary iterative development and assessment involving individual students working on it while thinking aloud to convey their thought processes, the QuILT was administered to students in the second semester of a full-year junior-senior level QM course. It strives to build on students' prior knowledge, actively engaging them in the learning process and helping them build connections between the abstract formalism and conceptual aspects of quantum physics without compromising the technical content. To assess the effectiveness of the QuILT, a pre-test and a post-test related to the addition of angular momentum were given to two classes of undergraduate students in a QM course at Pitt.

CONTENT SUMMARY

Here we first summarize the relevant angular momentum addition concepts. In QM, in which for every observable there is a corresponding Hermitian operator, the components of the orbital angular momentum operator \hat{L}_x , \hat{L}_y , and \hat{L}_z do not

commute with each other and therefore the components of orbital angular momentum are mutually incompatible observables. The operator corresponding to the square of the magnitude of the orbital angular momentum is \hat{L}^2 . In addition to the orbital angular momentum, \vec{L} , elementary particles, such as electrons, also possess intrinsic spin angular momentum, \vec{S} . The eigenvalues of the operator corresponding to the square of the magnitude of the spin angular momentum, \hat{S}^2 , are $s(s+1)\hbar^2$ where s is the spin quantum number (similar to the eigenvalues of \hat{L}^2 being $l(l+1)\hbar^2$). For a single electron, the spin quantum number, s , is $1/2$ and the value of m_s , the spin quantum number for the z -component of spin, are $\pm 1/2$. Because \hat{S}^2 and \hat{S}_z commute, we use the quantum numbers s and m_s to denote their simultaneous eigenstates as $|s, m_s\rangle$.

The addition of angular momentum formalism is applicable to the addition of two orbital angular momenta, two spin angular momenta or a combination of orbital and spin angular momenta. Here, we discuss the formalism only for spin degrees of freedom. If a quantum system contains two particles with individual spin angular momentum quantum numbers s_1 and s_2 , the total spin angular momentum quantum number of the system can range from $s_1 + s_2$ down to $|s_1 - s_2|$, i.e., $s = s_1 + s_2, s_1 + s_2 - 1, \dots, |s_1 - s_2|$. The z -component of the total spin angular momentum of the system equals the sum of the z -components of the spin angular momenta of the individual particles, i.e., $m_s = m_{s_1} + m_{s_2}$.

For a system consisting of two spin-1/2 particles, there are two common ways to represent the basis

vectors for the product space. Since the spin quantum numbers $s_1 = 1/2$ and $s_2 = 1/2$ are fixed, we can use the “uncoupled representation” and express the orthonormal basis vectors for the product space as direct products $|s_1, m_1\rangle \otimes |s_2, m_2\rangle \equiv |m_1\rangle \otimes |m_2\rangle$.

In this uncoupled representation, the operators related to each particle (subspace) act on their own states, e.g.,

$$\hat{S}_{1z}|1/2\rangle_1 \otimes |-1/2\rangle_2 = \frac{\hbar}{2}|1/2\rangle_1 \otimes |-1/2\rangle_2 \quad \text{and}$$

$$\hat{S}_{2z}|1/2\rangle_1 \otimes |-1/2\rangle_2 = -\frac{\hbar}{2}|1/2\rangle_1 \otimes |-1/2\rangle_2 .$$

On the other hand, we can use the “coupled representation” and work with states of total spin quantum number. The total spin for the two spin-1/2 particle system, s , is either $1/2 + 1/2 = 1$ or $1/2 - 1/2 = 0$. When the total spin quantum number s is 1, the quantum number m_s

for the z -component of the total spin, S_z , can be 1, 0, and -1. When the total spin quantum number s is 0, m_s can only be 0. Therefore, the basis vectors for $s=1$

in the coupled representation are $|s=1, m_s=1\rangle$, $|s=1, m_s=0\rangle$, $|s=1, m_s=-1\rangle$ and $|s=0, m_s=0\rangle$.

Thus, in the coupled representation, the state of a two-spin system is not a simple product of the states of each individual spin although we can write each coupled state as a linear superposition of a complete set of uncoupled states. Below we describe some of the relevant difficulties that can hinder student learning of the addition of angular momentum.

INVESTIGATION OF DIFFICULTIES

Difficulty 1: Confusion between Hilbert space and physical space

The dimension of a Hilbert space is equal to the number of linearly independent basis vectors, e.g., the number of linearly independent eigenstates of any operator that acts on the states in that space. For example, for a particle in a one dimensional (1D) infinite square well, the infinitely many energy eigenstates, $|\psi_n\rangle$, corresponding to the Hamiltonian operator, form a complete set of basis vectors for the infinite dimensional Hilbert space. We find that many students were confused about the dimensions of the Hilbert space and the physical space [3]. The following multiple choice question was given to 33 students to probe whether they could distinguish between the one-dimensional physical space in which the particle is confined and the infinite dimensional Hilbert space of system states.

- Choose all of the following statements that are

correct for a particle interacting with a one dimensional (1-D) infinite square well.

- (1) The appropriate Hilbert space for this system is one-dimensional.
- (2) The energy eigenstates of the system form a basis in a 1-D Hilbert space.
- (3) The position eigenstates of the system form a basis in a 1-D Hilbert space.

A. none B. 1 only C. 2 only D. 3 only E. all

The Hilbert space for the system in which the state of the system lies is infinite-dimensional, while the physical space in which the particle is confined is one-dimensional. Only 48% of the students chose the correct option A. Most other students incorrectly believed that the Hilbert space for the system is one-dimensional (approximately 25% selected option E).

Difficulty 2: Incorrectly calculating the dimension of a product space by adding the dimensions of the subspaces

The Hilbert space corresponding to the spin angular momentum of a single spin-1/2 particle is two-dimensional. For example, the z -component of the spin of an electron has two eigenstates, $|s=1/2, m_s=1/2\rangle$ and $|s=1/2, m_s=-1/2\rangle$ (or $|m_s=1/2\rangle$ and $|m_s=-1/2\rangle$ for short, since $s=1/2$ is a fixed number for an electron). If a system consists of two electrons, the product space corresponding to the spin degree of freedom will be four-dimensional, which is the product of the dimensions of the Hilbert spaces of each of the spins separately. The basis vectors of the four dimensional product space in the uncoupled representation are $|m_{s1}=1/2\rangle \otimes |m_{s2}=1/2\rangle$, $|m_{s1}=1/2\rangle \otimes |m_{s2}=-1/2\rangle$, $|m_{s1}=-1/2\rangle \otimes |m_{s2}=1/2\rangle$ and $|m_{s1}=-1/2\rangle \otimes |m_{s2}=-1/2\rangle$.

We find that students in general have great difficulty finding the dimension of a product space containing two or more angular momenta. When asked about the dimension D of a product space consisting of two subspaces of dimensions D_1 and D_2 , many students incorrectly believed that $D = D_1 + D_2$ instead of $D_1 \times D_2$. Discussions with individual students suggest that such a misconception often originates from the fact that the formalism is called “addition” of angular momentum (some students take the word addition literally) or the fact that the simplest example which helps students learn about the product space is for two spin-1/2 particles. In this case, the dimension of the product space is four, which equals 2×2 but is also $2 + 2$. When we asked 11 students

in an upper-level undergraduate QM course about the dimension of the product space for a system containing two spin-1 particles, four of them provided the incorrect answer $6 = 3 + 3$ instead of the correct answer $9 = 3 \times 3$. It appears, therefore, that using the example of two two-dimensional spaces has a disadvantage in terms of these issues. However, due to its simplicity, that is the most popular choice to illustrate issues related to the addition of angular momentum.

Difficulty 3: Difficulty in choosing a convenient basis to represent an operator as an $N \times N$ matrix in an N -dimensional product space

Students often have difficulty, e.g., in figuring out when it would be convenient to choose the basis vectors for the product space to be in the coupled or uncoupled representations. Many have difficulty representing an operator in matrix form. For example, when 26 undergraduate students were asked to choose a basis for two spin-1/2 particles and write the matrix corresponding to the operator $\hat{S}_1 \cdot \hat{S}_2 = (\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2)/2$ in that basis, 15% of the students could not find a complete set of basis vectors for the product space. Moreover, those who chose the uncoupled representation often had difficulty figuring out how to write \hat{S}^2 in a matrix form even though they were given the appropriate Clebsch-Gordon Coefficient (CGC) table to write the coupled states in terms of uncoupled states and vice versa. About 1/3 of the students did not realize that the basis vectors in the coupled representation are eigenstates of the operator \hat{S}^2 , so that the matrix elements of $\hat{S}_1 \cdot \hat{S}_2$ can be calculated more easily in the coupled representation than in the uncoupled representation. Some students mistakenly thought that the basis vectors in the product space are simply a collection of the basis vectors for the subspaces. For example, for the two spin-1/2 particle system, about 8% of the students incorrectly wrote down the basis vectors as $|s_1 = 1/2, m_1 = 1/2\rangle$, $|s_1 = 1/2, m_1 = -1/2\rangle$, $|s_2 = 1/2, m_2 = 1/2\rangle$, and $|s_2 = 1/2, m_2 = -1/2\rangle$ and constructed incorrect 2×2 matrices for the operators in the product space that they were asked to write in the matrix form in their chosen basis.

Difficulty 4: Incorrectly believing that the dimension of a matrix representing an operator in a given basis depends on the choice of basis vectors

The dimension of the product space is independent of the basis or representation chosen. For example, both the uncoupled and coupled representations for the

two spin-1/2 particle system have four basis vectors since the product space is four-dimensional. Several students displayed an inconsistency in interpreting the dimension of the product space depending upon the basis chosen. For example, 5 out of 11 undergraduate students incorrectly believed that the matrix for the operator $\hat{S}_{1z} + \hat{S}_{2z}$ is two-dimensional in the uncoupled representation. However, when explicitly asked about the same operator in the coupled representation, they could correctly construct a 4×4 diagonal matrix with the eigenvalues of $\hat{S}_{1z} + \hat{S}_{2z}$ in the diagonal positions using the basis vectors $|1,1\rangle$, $|1,-1\rangle$, $|1,0\rangle$ and $|0,0\rangle$. Discussions with individual students suggest that some of them were unclear about the fact that the dimension of the product space should always be equal to the number of linearly independent vectors in that space and does not depend on the choice of basis vectors.

DEVELOPMENT OF QUILT

We developed a QuILT to help reduce the difficulties faced by students in learning about angular momentum addition. The QuILT builds on the prior knowledge of students and uses a guided inquiry-based approach in which various concepts build on each other gradually. It was developed based on the difficulties found in written surveys given to various undergraduate and graduate students in QM classes and via individual interviews with a subset of students. The QuILT is accompanied by warm-up activities that students can work on as homework that helps them review the fundamental concepts related to angular momentum in QM. The development of the QuILT went through a cyclical interactive process which included the following stages: development of the preliminary version based on a theoretical analysis of the underlying knowledge structure and research on student difficulties; implementation and evaluation of the QuILT by administering it individually to students and asking them to think aloud while they worked on it; determining its impact on student learning and assessing what difficulties remained; and refinements and modifications based on the feedback from the implementation and evaluation.

The QuILT has two parts: one part is related to the uncoupled representation and the other to the coupled representation. As noted earlier, before working on the QuILT, students work on warm-up exercises which help them learn the basics of angular momentum. Here, we only discuss the section related to the dimension of the product space in the first part of the QuILT. At the beginning of the first part, students are asked about the dimension of the product space for the

two spin-1/2 particle system. Together with the correct answer that the dimension is $4 = 2 \times 2$, a distractor $4 = 2 + 2$ is given. This strategy forces students to notice that these two choices are different and learn about the dimension of the product space by discussing their answers with peers. We note that the QuILT can be used as a self-study tool, but when students work on them in class, peer discussion is exploited throughout. After the prediction, students first go through a guided approach to constructing the basis vectors in the uncoupled representation for two spin-1/2 particles and learn about the fact that the operators \hat{S}_{1z} and \hat{S}_{2z} only act on their respective subspaces in the uncoupled representation.

After help in constructing basic understanding about the uncoupled representation, students are given the following multiple-choice question:

- Choose all the statements that are correct.
- (1) $|1/2\rangle_1 \otimes |1/2\rangle_2$ is an eigenstate of \hat{S}_{1z} and \hat{S}_{2z} but not \hat{S}_1^2 or \hat{S}_2^2 .
 - (2) $|1/2\rangle_1 \otimes |1/2\rangle_2$ is an eigenstate of \hat{S}_{1z} , \hat{S}_{2z} , \hat{S}_1^2 and \hat{S}_2^2 .
 - (3) $|1/2\rangle_1 \otimes |1/2\rangle_2$ is an eigenstate of \hat{S}_{1z} , \hat{S}_{2z} , \hat{S}_1 and \hat{S}_2 .

Students discuss their answers with peers (correct answer is (2) only) and learn that in the uncoupled representation, the basis vectors are eigenstates of the individual spin operators \hat{S}_{1z} , \hat{S}_{2z} , \hat{S}_1^2 and \hat{S}_2^2 . They also learn to calculate the individual matrix elements and construct various matrices for the operators in the uncoupled representation (in the first part of the QuILT). For example, for the operator $\hat{S}_{1z} + \hat{S}_{2z}$, students learn that $(\hat{S}_{1z} + \hat{S}_{2z})|1/2\rangle_1 \otimes |-1/2\rangle_2 = 0$. They are guided to conclude that one matrix element is ${}_2\langle -1/2 | \otimes {}_1\langle 1/2 | (\hat{S}_{1z} + \hat{S}_{2z}) | 1/2\rangle_1 \otimes |-1/2\rangle_2 = 0$ and then they practice finding the other matrix elements.

In order to generalize their understanding of the product space to more complicated situations, students are later asked to consider the product space for a system of three spin-1/2 particles in the uncoupled basis. The following question explicitly asks for the dimension of the Hilbert space:

- What is the dimensionality of the spin space of a three spin-1/2 system?

(a) 2 (b) $2 + 2 + 2 = 6$ (c) $3^2 = 9$ (d) $2^3 = 8$

Here, students are given an opportunity to think about and discuss with peers the fact that the dimension of a product space is the product of the dimensions of the subspaces (and hence the correct answer is (d)). They

are further asked to construct a complete set of eight basis vectors and then calculate several diagonal and off-diagonal matrix elements of the operator $\hat{S}_{1z} + \hat{S}_{2z} + \hat{S}_{3z}$ in the uncoupled representation.

PRELIMINARY EVALUATION

We designed a pre-test and a post-test to assess the issues related to the addition of angular momentum. The pre-test was administered to 26 undergraduate students in the 2009 spring semester and 2010 spring semester after traditional instruction and the post-test was administered to the same two groups of students after they had worked on the QuILT. The questions in the pre-test and the post-test were similar but used product spaces for quantum systems with different spin. In particular, in the pre-test, the system contained two spin-1/2 particles, while the system in the post-test had one spin-1/2 particle and one spin-1 particle. In both the pre-test and post-test, students had to choose a convenient basis (coupled or uncoupled) to express the operators $\hat{S}_{1z} + \hat{S}_{2z}$ and $\hat{S}_1 \cdot \hat{S}_2$. The average correctness percentage for all questions in the pre-test was 40% in 2009 (with 9 students) and 27% in 2010 (with 17 students). After the students learned the topic using the QuILT, their post-test average score was 73% and 71% in 2009 and 2010, respectively, a statistically significant improvement. (The p-value of the t-test for the pre/post test scores of all the 26 students is less than 0.001.)

SUMMARY

We find that students have many common difficulties related to the addition of angular momentum. For example, many students were unclear about the dimension of the product space. Based upon the investigation of students' difficulties, we developed a research-based QuILT to improve students' understanding of the addition of angular momentum. Our preliminary evaluation shows that the QuILT helps improve students' understanding of concepts related to the addition of angular momentum.

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