

Adapting a Theoretical Framework for Characterizing Students' Use of Equations in Physics Problem Solving

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Abstract. Previous studies have focused on the resources that students activate and utilize while solving a given physics problem. However, few studies explore how students relate a given resource such as an equation, to various types of physics problems and contexts and how they ascertain the meaning and applicability of that resource. We explore how students view physics equations, derive meaning from those equations, and use those equations in physics problem solving. We adapt Dubinsky and McDonald's description of APOS (action-process-object-schema) theory of learning in mathematics, to construct a theoretical framework that describes how students interpret and use equations in physics in terms of actions, processes, objects, and schemas. This framework provides a lens for understanding how students construct their understanding of physics concepts and their relation to equations. We highlight how APOS theory can be operationalized to serve as a lens for studying the use of mathematics in physics problem solving.

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INTRODUCTION

Problem solving is an important area of research in physics education. Research in this area has converged on the consensus that most novice students tend to use means-ends strategies for problem solving[1]. As opposed to experts who typically start with a qualitative conceptual analysis of the problem, novices often seek an equation or formula in which they can plug in known quantities to solve for the quantity asked for in the problem.

Prior research on the use of equations and other mathematical forms in physics was completed by Tuminaro [2] who constructed a theoretical framework to describe how students understand and use mathematics in physics. This framework integrated the ideas of diSessa's p-prims [3], Sherin's symbolic forms [4], Collins' and Ferguson's epistemic games [5], and frames [6]. Based on this framework, Tuminaro identified four different kinds of errors that students made while using math in physics problems: (i) cueing an inappropriate math resource, (ii) inappropriately mapping a math resource, (iii) making inappropriate moves within an epistemic game, and (iv) playing an inappropriate game by incorrectly framing a problem situation.

Reflecting on reasons (i) and (ii) cited by Tuminaro, as well as the literature on physics

problems solving which shows novices' preference for equation seeking, we conclude that a common mathematical resource that novice students tend to cue, often inappropriately, to solve a physics problem, is an equation in which to plug in numerical values.

Sherin [7] completed research on students' use of equations in physics in terms of symbolic forms – a structure that associates a conceptual schema with a pattern of symbols in an equation. While this approach is a productive way of interpreting how students might think about equations, we also believe that since equations are basically mathematical constructs, it would be worthwhile examining other ways in which the mathematics education research community has characterized students' understanding of mathematical concepts.

A theoretical framework that is often used for characterizing student understanding of mathematical concepts is the APOS (action-process-object-schema) framework proposed by Dubinsky and McDonald [8]. The purpose of this paper is to demonstrate how the APOS framework can be adapted to characterize student understanding and use of equations in physics problem solving. We briefly discuss how this adaptation is consistent with problem solving literature. We conclude this paper by discussing concerns regarding the validity of this approach as well as implications for interdisciplinary research.

TABLE 1. Description of the original APOS levels as per Dubinsky and McDonald [8]

Level	Description
A (action)	“An <i>action</i> is a transformation of objects perceived by the individual as essentially external and as requiring, either explicitly or from memory, step-by-step instructions on how to perform the operation.”
P (process)	“When an action is repeated and the individual reflects upon it, he or she can make an internal mental construction called a <i>process</i> which the individual can think of as performing the same kind of action, but no longer with the need of external stimuli. An individual can think of performing a process without actually doing it, and therefore can think about reversing it and composing it with other processes.”
O (object)	“An <i>object</i> is constructed from a process when the individual becomes aware of the process as a totality and realizes that transformations can act on it.”
S (schema)	“... a <i>schema</i> for a certain mathematical concept is an individual’s collection of actions, processes, objects, and other schemas which are linked by some general principles to form a framework in the individual’s mind that may be brought to bear upon a problem situation involving that concept. This framework must be coherent in the sense that it gives, explicitly or implicitly, means of determining which phenomena are in the scope of the schema and which are not.”

APOS FRAMEWORK IN MATHEMATICS

APOS is a framework that has been used in mathematics education research to characterize students’ understanding and use of mathematical constructs, such as functions. While APOS is not a theory of learning, it shares the hierarchical nature of theories by Piaget [9] and Bloom [10] in that it categorizes a learner into sequential levels of understanding. Table 1 describes each APOS level of understanding. We have quoted directly from Dubinsky and McDonald [8] so as not to embellish their descriptions. The use of APOS to characterize student understanding of mathematical functions [11], may provide insights toward understanding how APOS may be adapted to characterize student understanding of physics equations..

At the A level a student simply understands a function as a procedure for transforming one number, the input, into another number, the output. At the P level a student sees a function as a trend. They do not need to do explicit calculations to predict what would happen to the output if you changed the input in a particular way. At the O level a function is a mathematical entity which has properties, e.g. continuous or discontinuous, with which mathematical actions such as addition or multiplication can be performed. At the S level a student sees a function in relation to other mathematical objects. They can express a function in multiple representations, such as graphs or tables, and translate between them.

ADAPTING THE APOS FRAMEWORK

We propose that the APOS can be adapted for characterizing the specific use of mathematical

constructs in the context of physics. Here, we limit ourselves to discussing how the framework can be adapted to characterizing student use of equations in physics problem solving; however, we purport that the APOS can be adapted for other mathematical objects such as graphs or perhaps even free body diagrams that are even more specific to physics than equations.

Table 2 demonstrates the adaptation of APOS for physics equations. Each of the levels is subdivided into HIGH and LOW levels of performance.

BRIEF CONNECTION TO PROBLEM SOLVING LITERATURE

APOS characterizes students’ use of equations in physics in a way that is consistent with other literature on physics problem solving. For instance, the work overlaps with Chi et al [12] that novices tend to focus on surface features of a problem while experts tend to focus on deep structure. This is consistent with the notion that novices are more likely to function at the A (action) or P (process) levels. They focus on the individual variables in an equation and not on the overall physical meaning or its place in physics. Experts, on the other hand tend to focus on causal relationships and the physical principles underlying a problem, which is more consistent with being at the O (object) or S (schema) level.

More recently, however researchers [13] have begun to question the dichotomy in expert-novice ontology proposed by Chi. Rather they point out that what distinguishes an expert is their ability to seamlessly move across different ontological categories. Similarly, it is important to note that expertise is not defined by merely having an O or S level understanding. Rather an expert can seamlessly move across all four APOS levels depending upon the requirement of the problem.

TABLE 2. APOS framework adapted by us to characterize students' use of equations in physics

Level	Description
A (action)	A student sees the equation as a tool one can plug numbers into for known physical quantities to find out unknown physical quantity. LOW: Student can plug in numbers in one side of the equation, to calculate an unknown quantity on the other side: e.g. $x = x_0 + v_0t + \frac{1}{2}at^2$. A student given x_0 , v_0 , a and 't' can calculate 'x'. However, given all other quantities except one, she is unable to calculate that unknown quantity. HIGH: Student can calculate any of the quantities given all other quantities.
P (process)	A student sees the equation as a relationship that allows them to predict, without the need for explicit calculation, the behavior of a physical quantity given the behavior of another physical quantity. LOW: Student can predict what would happen to one of the physical quantities if other physical quantities were changed in a particular way e.g. given $F = Gm_1m_2/r^2$, a student would know that if you increased m_1 or m_2 F would increase, or if r were increased F would decrease, but in either case the student cannot predict more specifically whether F will double or quadruple. HIGH: Student would be able to explain more specifically what would happen e.g. if m_1 is doubled, then F is doubled, if 'r' is halved, then F is quadrupled.
O (object)	A student sees the equation as encapsulating a physical principle or situation and the conditions under which this equation is applicable. LOW: Student can associate an equation to a physical situation or principle: e.g. the student can recognize that the equation $x = x_0 + v_0t + \frac{1}{2}at^2$ represents motion under constant acceleration or that $F = ma$ is Newton's second law, but the student cannot articulate the conditions under which the equation is applicable. HIGH: Student can not only associate the equation with a physical situation or principle, but also articulate the underlying assumptions. For instance, the first equation above is applicable only if the acceleration is constant. The equation $F = ma$ is Newton's Law under the specific condition of constant mass.
S (schema)	A student sees the relationship between this equation and other equations related to the same physical phenomena, as well as to an entire class of physical phenomena such an equation might describe. LOW: Student can see how an equation is related to other equations that are also applicable in the same context. For instance, a student recognizes that the equation $x = x_0 + v_0t + \frac{1}{2}at^2$ is related to the equation $v = v_0 + at$ or the two combined lead to the equation: $v^2 = v_0^2 + 2a(x-x_0)$, but the student does not see how the equation might be related to a wider class of physical phenomena. HIGH: Student can see how an equation is not merely related to other equations typically used to describe the physical phenomena, but also equations in a wider class of physical phenomena. E.g. the equation $x = x_0 + v_0t + \frac{1}{2}at^2$ is the same equation as one used in projectile motion under constant gravity i.e. it can be extended from a one-dimensional motion to two-dimensional motion.

VALIDITY CONCERNS OF THE ADAPTED APOS FRAMEWORK

We presented our adaption of APOS to two experienced mathematics education researchers who were conversant with APOS. These individuals served as external reviewers for our work. Overall, the reviewers agreed with our adaptation, but they cited some concerns. The first two concerns were limitations of the original APOS theory. Addressing these limitations is beyond the scope of this paper.

1) APOS is purely an evaluative framework, not a developmental framework. There is no description of how the different stages derive from each other. For example, once one has identified that a student has a 'process' concept of motion from constant acceleration, how do we move this student to an 'object' conception?

2) APOS is somewhat inaccurate as a linear progression. Students are certainly capable of imagining functions (for example), where they can imagine a value being calculated even if they have no

idea how to do it. This is a process conception without an action conception. Creating a graph by walking in front of a motion sensor might be an example of this.

The reviewers completely agreed with our adaptations of A (action) and P (process) levels. They had the following disagreements about our descriptions of the O (object) and S (schema) levels.

3) The object description in our adaptation is not entirely consistent with the original description. An object is an entity that can be acted upon and transformed. We have connected the equation to a physical principle, but that is more of a schema conception (making connections) than an object conception (transforming the object). An object conception would be closer to an idea such as: How does the equation change if velocity is constant? How does the equation change if acceleration is not constant? How does the equation change if we change reference points or coordinate systems? What makes an equation an object is that the equation itself is being taken as an entity to be acted upon?

4) Along the lines above, our description of high schema may in fact be closer to an object concept.

Schemas have been vaguely described by Dubinsky and McDonald [8]. One could think of schema as a sense of the equation's 'place' in physics. How does it derive from lower level physical laws? What other equations or principles connect to or derive from this equation? Why is the constant acceleration model important? Based on these recommendations we have modified the O- and S levels in Table 3.

TABLE 3. Modified descriptions based on comments from external reviewers.

O	A student is able to see how an equation will be transformed when the conditions are changed. For instance, how will the equation $x = x_0 + v_0t + \frac{1}{2}at^2$ change if the acceleration were not constant? How would it change if the coordinate system were changed?
S	A student understands an equation's place in physics. Why it is relevant and in what situations? How does it derive from physical laws? What other equations or principles connect to or derive from this equation?

IMPLICATIONS

The adapted APOS framework provides another way of examining student understanding and use of equations in physics. As pointed out earlier, this adaptation can possibly be expanded beyond its application to equations. For instance, students understanding of other mathematical entities used in physics such as graphs or perhaps even vector/phasor diagrams may be amenable to being characterized in terms of an adapted APOS framework.

One application of this adapted framework is in the design of assessments of the level of students' understanding of a mathematical concept used in the context of physics. Because our adaptation is based on a framework used in mathematics education research, it affords the opportunity to compare students' understanding of concepts in mathematics with equivalent concepts in physics. For instance, a research program may investigate how students move across the APOS hierarchy with their understanding of a mathematics concept when they begin to use this concept in a physics course following the math course.

SUMMARY

Physics education research is inextricably linked with research on learning in mathematics. The two fields have the potential to inform each other. In that vein, we have demonstrated how theoretical framework in math education research can be adapted to characterize the way in which students understand equations in physics. To test the validity of our

adaptation, we presented our adaptation to two mathematics education researchers who agreed with our descriptions but cited concerns that we have addressed. We envisage the promise of fruitful interdisciplinary collaborations with mathematics education researchers based on this framework.

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